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Lagrangian submanifolds of complex space forms: parallelity conditions and curvature inequalities

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Preface

> *Manifolds are a bit like pornography: hard to define, but you know one when you see one.*
>
> – Shmuel Weinberger

This thesis was written in the academic year 2014-2015 at KU Leuven, in fulfilment of the requirements for the degree of Master of Science in Mathematics. The goal of the thesis is to study Lagrangian submanifolds of complex space forms. In the first part we consider constraints on the submanifolds that are related to the second fundamental form and therefore called parallelity conditions. In the second part, we apply delta-invariants on the submanifolds in order to attain curvature inequalities, which we use to study the (non-)immersibility of manifolds as Lagrangian submanifolds.

First and foremost, I would like to thank my supervisor, professor Joeri van der Veken, for guiding me on my travels through the lands of Lagrangian submanifolds and for giving me a lot of freedom and plenty of help in writing this thesis. Furthermore I owe much thanks to professors Paul Igodt and Luc Vrancken for going through the effort of reading this thesis and for their interest in my work. I am very grateful to dr. Bart Dioos for always being welcome in his office to ask questions. Thanks to professors Johan Deprez, Antoine van Proeyen and Marco Zambon for providing me with some information I required. Finally, thanks to the entire geometry section at KU Leuven for all the seminars and discussions the past year, it has been a pleasure!

Abstract

In this thesis, we study Lagrangian submanifolds. These submanifolds play a fundamental role in symplectic geometry as well as in (complex) Riemannian geometry. A major difference is that where a local classification is trivial in symplectic geometry, it is not in Riemannian geometry. Such classification, however, is far from complete.

Preliminaries

In the first two chapters of this thesis we give some preliminaries that we will use throughout. We have opted to give the reader an elaborate reminder of the most important notions, structures and their properties in Riemannian geometry and submanifold theory. Special focus was placed on the so-called *complex space forms*, manifolds with a complex structure J that has several nice properties, and constant holomorphic sectional curvature. They play the role of ambient manifold in this thesis. In particular, we will always be working in one of three complete, simply connected complex space forms: the complex Euclidean space $\mathbb{C}^n$, the complex projective space $\mathbb{C}P^n$ or the complex hyperbolic space $\mathbb{C}H^n$.

The objects of study are the *Lagrangian submanifolds* of these complex space forms. They are the totally real submanifolds of maximal dimension, i.e. the submanifolds for which J provides an isometry between the tangent and normal bundle. We give their basic properties, define a canonical basis of the tangent space which we will need further in the thesis, and investigate the Cartan structure equations.

Part I

The first part of this thesis is devoted to parallelity conditions. The first chapter introduces the notions of parallelity we work with, in a global setting as well as in a Lagrangian setting. We impose these conditions on three important notions: the *mean curvature* H, the *second fundamental form* h and the *cubic form* C. We study conditions common in submanifold theory, such as being totally geodesic, totally umbilical, parallel, ... However, we also study newer conditions more suitable for Lagrangian submanifolds that have previously not been investigated: the condition of having *pseudo-parallel cubic form*, which was suggested in [DVV09], and we introduce a completely new condition, that of H-pseudo-parallelity.

The second chapter of part I uses the canonical basis to give a decomposition of the

tangent space at a point. We base ourselves on techniques that were applied in [Eji82] to study minimal Lagrangian submanifolds of constant sectional curvature, and in [Dil+12] to classify parallel Lagrangian submanifolds in $\mathbb{C}P^n$, to name a few examples.

In the third and final chapter of the first part, we give some classification results that were achieved in the domain of Lagrangian submanifolds: the classification of Lagrangian surfaces of constant curvature, the H-umbilical submanifolds and the parallel submanifolds in $\mathbb{C}P^n$.

Part II

The second part of this thesis is about Chen's *δ-invariants.* These are obtained by taking the scalar curvature τ of a manifold and "throwing away" some sectional curvatures. The first chapter of part II introduces these invariants, gives an interpretation for some special cases, and gives a general optimal inequality between the δ-invariants and the squared mean curvature. In particular, we find a nice inequality which holds for totally real submanifolds of complex space forms, and thus for Lagrangian submanifolds. Some corollaries obtained from this inequality are given, mostly in relation with the scalar curvature τ and the Ricci curvature Ric.

The second chapter is about the inequality restricted to Lagrangian submanifolds. Whereas the inequality was optimal when considering all submanifolds, it is no longer optimal when we only look at Lagrangian submanifolds: if equality is satisfied, then the submanifold is minimal. We have worked out the proof of an improved, optimal inequality for Lagrangian submanifolds given in [Che+13] in full detail.

The last chapter is a collection of corollaries of this improved optimal inequality. Using concepts from topology, we give a vanishing theorem for the mean curvature of certain compact Lagrangian submanifolds, and combined with δ-invariants this leads to a non-immersibility theorem. We also provide improvements of inequalities given in the first chapter of part II.

List of symbols

A	Shape operator
B^k	Exact k-forms
C	Cubic form
H^k_{dR}	k-th de Rham cohomology group
H	Mean curvature
J	Almost complex structure
K	Sectional curvature
$M^n(4c)$	Complex space form of constant holomorphic sectional curvature $4c$
$M^n(c)$	Real space form of constant sectional curvature c
M^n	Riemannian manifold of dimension n
R	Riemann-Christoffel curvature tensor
TM	Tangent bundle to manifold M
$T^\perp M$	Normal bundle to submanifold M
T_pM	Tangent space to manifold M at point p
$T^\perp_p M$	Normal space to submanifold M at point p
UM	Unit vectors of tangent bundle to manifold M
U_pM	Unit vectors of the tangent space to manifold M at point p
Z^k	Closed k-forms
$[.,.]$	Lie bracket
Λ^k	k-forms
Λ^k_p	k-forms at a point p
$\hat{M}$	Universal Riemannian covering space of a manifold M
Ω^k	Smooth k-forms
Ω^i_j	Curvature form
Φ	Maslov form
Π	Hopf projection
$\bar{R}$	Curvature tensor of the Van der Waerden-Bortolotti connection
Ric	Ricci curvature / Ricci tensor
$R^\perp$	Normal curvature tensor
Ξ	Cyclic sum
δ_{ij}	Kronecker delta
δ	Delta-invariant
$\langle .,. \rangle$	Riemannian metric

∇	Levi-Civita connection
$\bar{\nabla}$	Van der Waerden-Bortolotti connection
$\nabla^{\perp}$	Normal connection
$\mathcal{F}(M)$	Set of smooth functions on M
$\mathcal{N}$	Nijenhuis tensor
ω^i	Dual form to vector of othonormal basis
ω_i^j	Connection form
π_1	Fundamental group
τ	Scalar curvature
$\wedge$	Exterior product of forms $(\omega \wedge \eta)$
$\wedge$	Wedge operator on vector fields $(X \wedge Y)$
b_k	k-th Betti number
d	Exterior derivative
h	Second fundamental form

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Introduction

The study of Lagrangian submanifolds originates from symplectic geometry. A $2n$-dimensional manifold $\tilde{M}^{2n}$ is called symplectic if it admits a closed, nondegenerate 2-form ω, the symplectic form. An n-dimensional submanifold M^n of a symplectic manifold $(\tilde{M}^{2n}, \omega)$ is called Lagrangian if $\omega|_M \equiv 0$. Due to Givental's theorem, a local classification of Lagrangian submanifolds is trivial from the symplectic point of view [Arn90; DS01; DS04].

Symplectic manifolds and their Lagrangian submanifolds appear naturally in many aspects of physics. For example, in classical mechanics and mathematical physics [Arn89; HZ11], but also in string theory [Wit95] and in supersymmetric field theories [GJ00].

From the Riemannian point of view, a local classification is far from trivial. The study of totally real submanifolds was initiated in the 70's [Bla75; CHL77; CO74a; Hou73; Kon76; LOY75a; LOY75b; Yan76; YK76a; YK76b]. It was quickly noticed that "when [they have half the dimension of the ambient space], totally real submanifolds have many interesting properties" [YK76a], which are exactly the Lagrangian submanifolds.

However, since research of Lagrangian manifolds has started, there is still no complete classification. We can approach the study of Lagrangian submanifolds in two ways: on the one hand, one can put additional constraints (in addition to being Lagrangian) on a manifold and attempt the classify these submanifolds. On the other hand, one can study when a given Riemannian manifold admits a Lagrangian isometric immersion. The structure of this thesis reflects this twofold approach.

In the first part of this thesis we study constraints involving the second fundamental form h. Concretely we will work with parallelity conditions on h itself and on its normalised trace H, the mean curvature. We will give the definitions of these constraints, explain the properties that follow and show how these condition interact.

The second part of this thesis deals with curvature inequalities. We give the definition of the (intrinsic) δ-invariants introduced by Chen [Che93; Che94; Che95] and their relation with the (extrinsic) mean curvature. We will focus on the optimal general inequality for δ-invariants. Once we restrict ourselves to Lagrangian submanifolds again, we show that the aforementioned inequality is no longer optimal [Che00a] but can be improved [Che+13]. Finally, we will give some corollaries of the improved inequality.

Chapter 1

Riemannian manifolds, complex manifolds and their submanifolds

This chapter serves as an introduction to Riemannian geometry, complex manifolds and submanifold theory. It is meant to remind the reader of the notions and the properties that are commonly used, as well as to provide the framework for this thesis. Special attention is given to complex space forms, as these will be the ambient spaces containing Lagrangian submanifolds we work with.

The contents of this chapter are based on standard works such as [Che11; DC92; Gra65; Mor01; YK85], as well as the lecture notes on Riemannian geometry and submanifold theory [Vraa; Vrab] by Luc Vrancken.

1.1 Riemannian manifolds and complex manifolds

Before we introduce the "Riemannian" aspect, let us first recall what a differentiable manifold and its tangent space are and how vectors and vector fields come in play.

Definition 1.1.1. A *differentiable manifold* M^n of dimension n consists of a set M, together with a collection of injective maps $\phi_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ (where U_α is an open subset of $\mathbb{R}^n$) such that

(i) $\bigcup_\alpha \phi(U_\alpha) = M$,

(ii) for every pair α, β with $V = \phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset$, we have that $\phi_\alpha^{-1}(V)$ and $\phi_\beta^{-1}(V)$ are open subset of $\mathbb{R}^n$ and the maps $\phi_\alpha^{-1} \circ \phi_\beta$ and $\phi_\beta^{-1} \circ \phi_\alpha$ are differentiable,

(iii) the family (U_α, ϕ_α) is maximal with respect to properties (i) and (ii).

We will call the family (U_α, ϕ_α) an *atlas*, and an element of the atlas is called a *chart*.

Definition 1.1.2. Let M^n be a differentiable manifold. A differentiable map $\alpha : I \to M$ is called a *curve* in M. Suppose that $\alpha(t_0) = p$ and let $\mathcal{F}(M)$ denote the set of differentiable functions on M. The *tangent vector* to the curve α in $t = t_0$ is the map

$$\alpha'(t_0) : \mathcal{F}(M) \to \mathbb{R} : f \mapsto \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=t_0}.$$

A tangent vector at a point p is the tangent vector to a curve α in t_0 where $\alpha(t_0) = p$. We identify two tangent vectors if they work in the same way on all locally differentiable functions. We denote the collection of all tangent vectors at a point p by T_pM, the *tangent space* to M at the point p. The (disjoint) union of the tangent spaces T_pM at all points p of M is called the *tangent bundle* TM.

Definition 1.1.3. A *vector field* X on a manifold M is a map that associates with every point p of M a tangent vector at the point p. We call X *differentiable* in a neighbourhood of p if and only if the map $X : M \to TM$ is differentiable in a neighbourhood of p. Moreover, we define

(i) $X(f)(p) = X(p)(f)$,

(ii) $(X + Y)(p) = X(p) + Y(p)$,

(iii) $(fX)(p) = f(p)X(p)$,

for any differentiable function f on a neighbourhood of p.

Property 1.1.4. Let X be a vector field an f and g differentiable functions on a neighbourhood of p. Then we have that

(i) $X(fg) = fX(g) + gX(f)$,

(ii) $X(af + bg) = aX(f) + bX(g)$.

Proposition 1.1.5. *If X and Y are differentiable vector fields, then there exists a unique differentiable vector field Z such that*

$$Z(p)(f) = X(p)(Yf) - Y(p)(Xf).$$

This vector field is called the Lie bracket *of X and Y and is denoted by $[X, Y] = XY - YX$.*

Property 1.1.6. Let X, Y, Z be differentiable vector fields, let f, g, h be differentiable functions and let a, b be real numbers. Then

(i) $[X, Y] = -[Y, X]$,

(ii) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$,

(iii) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$,

(iv) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

We now make the step from differential geometry to Riemannian geometry, by introducing Riemannian metrics and the Levi-Civita connection.

Definition 1.1.7. A *Riemannian metric* on a differentiable manifold M^n is a map that assigns to each $p \in M$ a positive definite, symmetric bilinear form $\langle .,. \rangle_p$ on T_pM such that for vector fields X, Y, the function $\langle X, Y \rangle(p) = \langle X(p), Y(p) \rangle_p$ is differentiable. If a differentiable manifold is equipped with a Riemannian metric, it is called a *Riemannian manifold*.

Remark 1.1.8. If X is a tangent vector, we will often write $\|X\|$ for the length of the vector X, i.e. $\|X\|^2 = \langle X, X\rangle$. Because the metric is positive definite, the norm is a positive function.

Definition 1.1.9. An *affine connection* ∇ on a differentiable manifold M is a map which associates to two differentiable vector fields X, Y a differentiable vector field $\nabla_X Y$ such that

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,

(ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,

(iii) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$,

where X, Y, Z are differentiable vector fields and f, g are real functions on M.

Proposition 1.1.10. *On a Riemannian manifold, there exists a unique affine connection that satisfies the following properties:*

(i) $X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ *(compatible),*

(ii) $\nabla_X Y - \nabla_Y X = [X, Y]$ *(symmetric),*

which is called the Levi-Civita connection*.*

From now on, the symbol ∇ will always be the Levi-Civita connection unless mentioned otherwise. We now let the Levi-Civita connection work on tensors rather than vector fields.

We say that a tensor T is of type (n, m) if it takes n vectors as arguments and returns m vectors.

Definition 1.1.11. If T is a tensor of type $(n, 1)$, we can define a tensor ∇T of type $(n + 1, 1)$, the "derivative" of T, by

$$(\nabla_X T)(Y_1, \ldots, Y_n) = \nabla_X T(Y_1, \ldots, Y_n) - \sum_{i=1}^{n} T(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_n),$$

and similarly for a tensor T of type $(n, 0)$ we define the tensor ∇T of type $(n + 1, 0)$ by

$$(\nabla_X T)(Y_1, \ldots, Y_n) = X(T(Y_1, \ldots, Y_n)) - \sum_{i=1}^{n} T(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_n).$$

We can then define the k-th order derivative by

$$(\nabla^k_{X_1,\ldots,X_k} T) = (\nabla_{X_1}(\nabla^{k-1}_{X_2,\ldots,X_k} T)).$$

Remark 1.1.12. This differentiation behaves well with respect to the metric: if T is a $(n, 1)$-tensor and we define the $(n + 1, 0)$-tensor

$$T'(Y_1, \ldots, Y_{n+1}) = \langle T(Y_1, \ldots, Y_n), Y_{n+1}\rangle,$$

then we find that

$$\langle(\nabla^k_{X_1,\ldots,X_k} T)(Y_1, \ldots, Y_n), Y_{n+1}\rangle = (\nabla^k_{X_1,\ldots,X_n} T')(Y_1, \ldots, Y_n, Y_{n+1}).$$

Definition 1.1.13. The *Riemann curvature tensor* R is defined by

$$\begin{aligned} R(X,Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \\ &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{\nabla_X Y} + \nabla_{\nabla_Y X} \\ &= \nabla^2_{X,Y} - \nabla^2_{Y,X}, \end{aligned}$$

and we interpret $R(X,Y)Z$ as a tensor of type (3,1).

Property 1.1.14. The curvature tensor R has the following properties:

(i) $R(X,Y) = -R(Y,X)$,

(ii) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$,

(iii) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$,

(iv) $\underset{X,Y,Z}{\Xi} R(X,Y)Z = 0$ (first Bianchi identity),

(v) $\underset{X,Y,Z}{\Xi} (\nabla_X R)(Y,Z) = 0$ (second Bianchi identity),

where Ξ denotes the cyclic sum over its indices.

A important operator closely related to the curvature operator is the following:

Definition 1.1.15. We define the *wedge operator* $\wedge$ as

$$(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y.$$

It satisfies all of the properties of the Riemannian curvature tensor mentioned above.

Definition 1.1.16. The *sectional curvature* K is defined by

$$K(X,Y) = \frac{\langle R(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

If π is a 2-dimensional subspace of T_pM spanned by vectors X, Y, we will also write $K(\pi) = K(X,Y)$. In fact, $K(\pi)$ is independent of the chosen basis.

For a surface S, the tangent space T_pS is 2-dimensional at every point. The sectional curvature $K(T_pM)$ equals the Gaussian curvature $K(p)$ at every point $p \in S$, hence they are both denoted by K. As a consequence, for a surface we have that the curvature operator takes the following form:

$$R(X,Y) = K(p)(X \wedge Y).$$

Definition 1.1.17. The *Ricci tensor* is defined as

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} \langle R(e_i, X)Y, e_i\rangle,$$

and the *Ricci curvature* of a unit vector X is defined as

$$\mathrm{Ric}(X) = \mathrm{Ric}(X,X) = \sum_{i=1}^{n} K(e_i, X),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM.

Definition 1.1.18. The *scalar curvature* is defined by

$$\tau = \sum_{i<j} K(e_i, e_j) = \frac{1}{2}\sum_{i,j} K(e_i, e_j) = \frac{1}{2}\sum_{i=1}^{n} \operatorname{Ric}(e_i),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM.

Definition 1.1.19. If $\operatorname{Ric}(X, Y)$ is a multiple of $\langle X, Y\rangle$ for each $X, Y \in T_pM$ and at every point $p \in M$, then we say that M is an *Einstein manifold.*

Definition 1.1.20. If $K(X, Y) = c$ with $c \in \mathbb{R}$, for all $X, Y \in T_pM$ and at every point $p \in M$, then we call M a *real space form* which we denote by $M(c)$. The curvature tensor of a real space form is of the form

$$R(X, Y) = c(X \wedge Y).$$

We say that M is a *flat* manifold when it is of constant sectional curvature $c = 0$.

Lemma 1.1.21 (Schur's Lemma). *Let M^n be a Riemannian manifold of dimension $n \geq 3$. Suppose that there is a function $c \in \mathcal{F}(M)$ such that at any point p and for any $X, Y \in T_pM$ we have that $K(X, Y) = c(p)$. Then c is actually a constant function, i.e. M is a real space form.*

We now turn towards Riemannian manifolds with complex structures.

Definition 1.1.22. A Riemannian manifold M is called an *almost complex manifold* if and only if it admits an *almost complex structure* J, i.e. for any $p \in M$, there exists a map $J_p : T_pM \to T_pM$ such that $J_p^2 = -\operatorname{Id}$ and for any X, $JX(p) := J_pX(p)$ is a differentiable vector field.

Remark 1.1.23. For M to have an almost complex structure, it is necessarily of even dimension. Because of this, when talking about an almost complex manifold M^n, n denotes the complex dimension rather than the real dimension.

Definition 1.1.24. Given a tensor field of type $(1, 1)$ on M, the *Nijenhuis tensor* of A is a tensor of type $(2, 1)$ given by

$$\mathcal{N}_A(X, Y) = -A^2[X, Y] + A[AX, Y] + A[X, AY] - [AX, AY].$$

Definition 1.1.25. We say that an almost complex manifold M is a *complex manifold* if the Nijenhuis tensor $\mathcal{N}_J$ of the almost complex structure J vanishes everywhere.

Definition 1.1.26. An almost complex manifold M for which J is *compatible* with the metric, i.e.

$$\langle X, Y\rangle = \langle JX, JY\rangle,$$

is called an *almost Hermitian manifold.* If M is moreover a complex manifold, it is called a *Hermitian manifold.*

Note that this compatibility is equivalent to $\langle X, JX\rangle = 0$. So we can see J as a rotation by 90 degrees: by repeatedly applying J to a vector, we obtain the sequence $X, JX, -X, -JX, X, JX, \ldots$. All these vectors lie in the plane spanned by the orthogonal vectors X and JX.

Definition 1.1.27. An almost Hermitian manifold M is called a *Kähler manifold* if $\nabla J = 0$, i.e. for all $X, Y \in T_pM$,

$$(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y = 0.$$

Remark 1.1.28. This conditions implies that $\mathcal{N}_J$ vanishes. Thus Kähler manifolds are always Hermitian manifolds.

1.2 Complex space forms

We ended the previous section by introducing Kähler manifolds. Suppose we want more symmetry on the manifold, then we could require that a Kähler manifold has constant sectional curvature c. However, the only Kähler manifolds of constant sectional curvature are either 2-dimensional or flat, which is too restrictive. Therefore we define the following notion:

Definition 1.2.1. The *holomorphic sectional curvature* of a Kähler manifold M is defined as $K(X, JX)$.

Rather than requiring a Kähler manifold to have constant sectional curvature, we impose that it must have constant holomorphic sectional curvature.

Definition 1.2.2. A Kähler manifold of constant holomorphic sectional curvature $4c$ is called a *complex space form*, which we denote by $M^n(4c)$.

The reason for the factor 4 is purely aesthetic, as will become clear soon. It is simply there to prevent having to write division by 4 constantly, and serves as a good way to distinguish real space forms and complex space forms. Analogous to real space forms, the curvature tensor of complex space forms takes a special form:

$$R(X, Y) = c\left((X \wedge Y) + (JX \wedge JY) + 2\langle X, JY\rangle J\right). \tag{1.2.1}$$

Property 1.2.3. This curvature tensor R of a complex space form J interacts with the complex structure J in the following ways:

(i) $R(X, Y) = R(JX, JY)$,

(ii) $R(X, JY) = -R(JX, Y)$,

(iii) $R(X, Y)J = JR(X, Y)$,

(iv) $\langle R(X, Y)JZ, JW\rangle = \langle R(X, Y)Z, W\rangle$,

(v) $\langle R(X, Y)JZ, W\rangle = -\langle R(X, Y)Z, JW\rangle$.

Proposition 1.2.4. *Let $M^n(4c)$ be a complex space form. Then the Ricci tensor of M is given by*

$$\mathrm{Ric}(X, Y) = 2c(n + 1)\langle X, Y\rangle, \tag{1.2.2}$$

and therefore M is an Einstein manifold.

Proof.

$$\begin{aligned}
\mathrm{Ric}(X,Y) &= \sum_{i=1}^{2n} \langle R(e_i,X)Y, e_i\rangle \\
&= c\sum_{i=1}^{2n} \left(\langle (e_i \wedge X)Y, e_i\rangle + \langle (Je_i \wedge JX)Y, e_i\rangle + 2\langle e_i, JX\rangle\langle JY, e_i\rangle\right) \\
&= c\sum_{i=1}^{2n} \left(\langle X,Y\rangle\langle e_i,e_i\rangle - \langle e_i,Y\rangle\langle e_i,X\rangle + \langle JX,Y\rangle\langle Je_i,e_i\rangle \right.\\
&\qquad \left. -\langle Je_i,Y\rangle\langle e_i,JX\rangle + 2\langle e_i,JX\rangle\langle JY,e_i\rangle\right) \\
&= c\left(2n\langle X,Y\rangle - \sum_{i=1}^{2n}\left(\langle X, \langle Y,e_i\rangle e_i\rangle + \langle JX, \langle JY,e_i\rangle e_i\rangle + \langle JX, \langle JY,e_i\rangle e_i\rangle\right)\right) \\
&= c\left(2n\langle X,Y\rangle - \langle X,Y\rangle + \langle JX,JY\rangle + 2\langle JX,JY\rangle\right) \\
&= 2c(n+1)\langle X,Y\rangle,
\end{aligned}$$

which proves the proposition. □

A simply connected complete complex space form can be identified with a complex projective space $\mathbb{C}P^n$, a Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n$ according as $c > 0$, $c = 0$ or $c < 0$.

The complex Euclidean space $\mathbb{C}^n$ is always given the Euclidean metric:

$$\langle X_p, Y_p\rangle = \mathrm{Re}(x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n).$$

Definition 1.2.5. The *complex projective space* $\mathbb{C}P^n$ can be defined as follows:

$$\mathbb{C}P^n = \mathbb{C}^{n+1}\backslash\{0\}/\{p \sim \lambda p \mid \lambda \in \mathbb{C}\backslash\{0\}\}.$$

Now, let $\Pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}P^n$ be the natural projection, and consider its restriction to $S^{2n+1}(1)$. From now on Π will always be the restriction map. This restriction is surjective and two points p, q have the same image if and only if $p = e^{it}q$ for some $t \in \mathbb{R}$. We give $\mathbb{C}^{n+1}\backslash\{0\}$ the Euclidean metric. Note that the position vector field P restricted to $S^{2n+1}(1)$ is a unit normal vector field on $S^{2n+1}(1)$, and thus $\xi = iP$ is a globally defined unit vector field on $S^{2n+1}(1)$. We obtain that $(d\Pi)_p$ is surjective and has kernel $\mathrm{span}\{\xi_p\}$ for any $p \in S^{2n+1}(1)$.

Now for any vector field X on $\mathbb{C}P^n$, there is a unique vector field $\tilde{X}$ on $S^{2n+1}(1)$ such that $(d\Pi)(\tilde{X}) = X$ and $\tilde{X}$ is everywhere orthogonal to ξ. We will call this vector field the *horizontal lift* of X. This allows us to put a metric on $\mathbb{C}P^n$:

Definition 1.2.6. The *Fubini-Study* metric on $\mathbb{C}P^n$ is the metric $\langle ., .\rangle_{FS}$ defined by

$$\langle X_p, Y_p\rangle_{FS} = \langle \tilde{X}_q, \tilde{Y}_q\rangle_{S^{2n+1}(1)},$$

where $\langle ., .\rangle_{S^{2n+1}(1)}$ is the metric on $S^{2n+1}(1)$ induced by the Euclidean metric of $\mathbb{C}^{n+1}$, and $q \in \Pi^{-1}(p)$.

Note that this definition is independent of the point q, because the map $p \mapsto e^{it}$ on $S^{2n+1}(1)$ is an isometry which preserves ξ and horizontal lifts from vectors fields on $\mathbb{C}P^n$.

The complex hyperbolic space $\mathbb{C}H^n$ can be defined in a similar way. We consider the space $\mathbb{C}_1^{n+1}$, which is the set $\mathbb{C}^{n+1}$ with metric

$$\langle X_p, Y_p \rangle = -x_0\bar{y}_0 + x_1\bar{y}_1 + \cdots + x_n\bar{y}_n.$$

Instead of the sphere $S^{2n+1}(1)$, we consider the anti-de Sitter space

$$H_1^{2n+1} = \{p \in \mathbb{C}_1^{n+1} \mid \langle p, p \rangle = -1\}.$$

We define the complex hyperbolic space $\mathbb{C}H^n$ as the set of equivalence classes of H_1^{2n+1} under the action $p \mapsto \lambda p$. Thus we find a projection $\Pi : H_1^{2n+1} \to \mathbb{C}H^n$, and we may define a metric on $\mathbb{C}H^n$ in the same way we did for the complex projective space.

1.3 Submanifold theory

In this section we will explain the idea of a submanifold and discuss the basic tools we have. We will also give commonly used constraints on submanifolds. For notational purposes, everything with a tilde $\sim$ will be related to the ambient manifold, e.g. $\tilde{M}, \tilde{R}, \tilde{\nabla}, \ldots$

Let M and $\tilde{M}$ be differentiable manifolds and let $\phi : M \to \tilde{M}$ be a differentiable map. However, this map only works on points of the manifold, but not on its tangent vectors. Therefore we have the derivative map:

Definition 1.3.1. Let $\phi : M \to \tilde{M}$ be a differentiable map between differentiable manifolds. Then at any point $p \in M$, consider the map

$$d\phi_p : T_pM \to T_{\phi(p)}\tilde{M} : v \mapsto \frac{d}{dt}(\phi \circ \gamma)\bigg|_{t=0},$$

where γ is a curve in M such that $\gamma(0) = p$ and $\gamma'(0) = v$.

Definition 1.3.2. Let $\phi : M \to \tilde{M}$ be a differentiable map between differentiable manifolds. We say that

(i) ϕ is an *immersion* if and only if $d\phi_p$ is injective for all $p \in M$,

(ii) ϕ is an *embedding* if and only if $\phi : M \to \phi(M)$ is a homeomorphism,

(iii) if ϕ is an immersion, then we will say (M, ϕ) is a submanifold of $\tilde{M}$.

In Riemannian geometry we require something stronger than an immersion.

Definition 1.3.3. Let $(M, \langle ., . \rangle_M)$ and $(\tilde{M}, \langle ., . \rangle_{\tilde{M}})$ be Riemannian manifolds and $\phi : M \to \tilde{M}$ an immersion. We call ϕ an isometric immersion if and only if

$$\langle X, Y \rangle_M = \langle d\phi(X), d\phi(Y) \rangle_{\tilde{M}}, \tag{1.3.1}$$

for every $X, Y \in T_pM$ at every point $p \in M$.

On the one hand, we can start with M and $\tilde{M}$ both having their own metric, and verifying if a given immersion between them is isometric. On the other hand we could also take M to be a differentiable manifold immersed by ϕ in a Riemannian manifold $(\tilde{M}, \langle .,. \rangle_{\tilde{M}})$ and give M the metric defined by (1.3.1), which we call the *induced metric*. Then M becomes a Riemannian manifold $(M, \langle .,. \rangle_M)$ and ϕ becomes an isometric immersion between $(M, \langle .,. \rangle_M)$ and $(\tilde{M}, \langle .,. \rangle_{\tilde{M}})$.

Because $d\phi$ is injective, we will simply "forget" to write the $d\phi$, by identifying $d\phi(T_pM)$ with T_pM. We will also write $T_p\tilde{M}$ rather than $T_{\phi(p)}\tilde{M}$ since an immersion is locally an embedding.

We can split up the tangent space of $\tilde{M}$ as $T_p\tilde{M} = T_pM \oplus T_p^\perp M$. Thus we can denote by $\bar{X}^\top$ and $\bar{X}^\perp$ the tangent and normal components of a vector field $\bar{X}$ respectively. This decomposition is unique. We shall denote elements of T_pM with Roman letters (e.g. X, Y, ...) and elements of $T_p^\perp M$ with Greek letters (e.g. ξ, η, ...).

The first tools to study submanifolds are the formulas of Gauss and Weingarten. Let ∇ be the Levi-Civita connection of M and $\tilde{\nabla}$ the Levi-Civita connection of $\tilde{M}$, let X, Y be tangent vector fields and ξ a normal vector field. We can decompose the vector fields $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X \xi$ into tangent and normal components as follows:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.3.2}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi. \tag{1.3.3}$$

Definition 1.3.4. We call h is the *second fundamental form* of the immersion, A is the *shape operator* and $\nabla^\perp$ is the *normal connection*. Note that both the second fundamental form and the shape operator are tensorial.

Property 1.3.5. These have the following properties:

(i) $h(X, Y) = h(Y, X)$,

(ii) $\langle A_\xi X, Y \rangle = \langle A_\xi Y, X \rangle$,

(iii) $\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle$,

(iv) $\nabla^\perp$ has the usual properties of a connection.

Definition 1.3.6. Since $\nabla^\perp$ is a connection we can define its associated curvature, the *normal curvature* $R^\perp$ by

$$R^\perp(X, Y) = [\nabla_X^\perp, \nabla_Y^\perp] - \nabla_{[X,Y]}^\perp.$$

We will define another connection:

Definition 1.3.7. The *Van der Waerden-Bortolotti connection* $\bar{\nabla}$ is defined as

$$\bar{\nabla} = \nabla \oplus \nabla^\perp.$$

This simply means we apply the Levi-Civita connection of M on a tangent vector field, and we apply the normal connection on a normal vector field. For a vector $\tilde{Y} \in T_p\tilde{M}$ this would mean that

$$\bar{\nabla}_X(\tilde{Y}) = \nabla_X Y^\top + \nabla_X^\perp Y^\perp.$$

Its differentiation works similarly, for example when used on the second fundamental form we obtain

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Denote by R and $\tilde{R}$ the curvature tensors of M and $\tilde{M}$ respectively, and by $R^\perp$ the normal curvature. The equations of Gauss, Codazzi and Ricci are then given by

$$(\tilde{R}(X,Y)Z)^\top = R(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y, \tag{1.3.4}$$

$$(\tilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{1.3.5}$$

$$(\tilde{R}(X,Y)\xi)^\perp = R^\perp(X,Y)\xi + h(A_\xi X, Y) - h(X, A_\xi Y), \tag{1.3.6}$$

or equivalently by

$$\langle \tilde{R}(X,Y)Z), W\rangle = \langle R(X,Y)Z, W\rangle - \langle h(Y,Z), h(X,W)\rangle + \langle h(X,Z), h(Y,W)\rangle, \tag{1.3.7}$$

$$\langle \tilde{R}(X,Y)Z, \xi\rangle = \langle (\bar{\nabla}_X h)(Y,Z), \xi\rangle - \langle (\bar{\nabla}_Y h)(X,Z), \xi\rangle, \tag{1.3.8}$$

$$\langle \tilde{R}(X,Y)\xi, \eta\rangle = \langle R^\perp(X,Y)\xi, \eta\rangle - \langle [A_\xi, A_\eta]X, Y\rangle. \tag{1.3.9}$$

Since $\langle \tilde{R}(X,Y)\xi, Z\rangle = -\langle \tilde{R}(X,Y)Z, \xi\rangle$, we omit $(\tilde{R}(X,Y)\xi)^\top$ as it would be equivalent to the equation of Codazzi.

Definition 1.3.8. The *mean curvature* H is defined as the normalised trace of the second fundamental form, i.e.

$$H = \frac{1}{n}\operatorname{trace} h = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM^n.

Based on the second fundamental form, shape operator and mean curvature, we can define the following constraints:

Definition 1.3.9. A submanifold M is called

- *totally geodesic* if $h \equiv 0$,
- *totally umbilical* if A_ξ is a multiple of the identity for any ξ,
- *minimal* if $H = 0$.

If $\tilde{M}$ is an almost complex manifold, we can define constraints related to the almost complex structure J:

Definition 1.3.10. A submanifold M of an almost complex manifold $\tilde{M}$ is called

- *holomorphic* if $J(T_pM) \subset T_pM$ and therefore $J(T_pM) = T_pM$,
- *totally real* if $J(T_pM) \subset T_p^\perp M$, i.e. $\langle JX, Y\rangle = 0$ for all X, Y tangent to M.

And finally if $\tilde{M}$ is a Kähler manifold, we have:

Definition 1.3.11. A submanifold M^n of a Kähler manifold $\tilde{M}^m$ is called

- *Kähler* if it is holomorphic, then M is itself a Kähler manifold,
- *Lagrangian* if it is totally real and $n = m$, or equivalently $J(T_pM) = T_p^\perp M$.

1.4 Forms and cohomology

In this section, we define forms on a differentiable manifold, and show how they induce a cohomology.

Definition 1.4.1. We define the set of k-forms $\Lambda_p^k M$ at $p \in M$ as

$$\Lambda_p^k M = \{\omega_p : (T_p M)^k \to \mathbb{R} \mid \omega \text{ multilinear and alternating}\}.$$

Property 1.4.2. We have the following properties:

(i) $\Lambda_p^0 M \cong \mathbb{R}$,

(ii) $\Lambda_p^1 M = (T_p M)^*$, the dual space of $T_p M$,

(iii) $\Lambda_p^n M \cong \mathbb{R}$,

(iv) $\Lambda_p^k M \cong \{0\}$ for all $k > n$,

(v) $\Lambda_p^k M$ is a real vector space of dimension $\binom{n}{k}$.

Definition 1.4.3. The exterior product $\wedge$ of forms works as follows:

$$\wedge : \Lambda_p^k M \times \Lambda_p^l M \to \Lambda_p^{k+l} M : (\omega_p, \eta_p) \mapsto (\omega \wedge \eta)_p,$$

where

$$(\omega \wedge \eta)_p(v_1, \ldots, v_n) = \sum_{\sigma \in S_{k+l}} \operatorname{sign}(\sigma)\omega_p(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot \eta_p(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

However, we want to work with forms on the entire manifold rather than just in a point.

Definition 1.4.4. We define the set of all k-forms on a differentiable manifold M as

$$\Lambda^k M = \bigcup_{p \in M} \Lambda_p^k M,$$

and we restrict this set to the forms that are smooth:

$$\Omega^k(M) = \{\omega : M \to \Lambda^k M : p \mapsto \omega_p \mid \omega_p \in \Lambda_p^k M \text{ and } \omega \text{ smooth}\},$$

where smoothness means that for any vector fields $X_1, \ldots, X_k$, $\omega(X_1, \ldots, X_k)(p) : M \to \mathbb{R}$ is a smooth function.

Remark 1.4.5. A 0-form $\omega \in \Omega^0(M)$ is just a smooth function $M \to \mathbb{R}$, so $\Omega^0 M = \mathcal{F}(M)$.

Definition 1.4.6. We define the exterior derivative d as

$$d : \Omega^k(M) \to \Omega^{k+1}(M) : \omega \mapsto d\omega,$$

where

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{k+1}). \end{aligned}$$

Property 1.4.7. This exterior derivative has the following properties:

(i) if $\omega \in \Omega^0(M) = \mathcal{F}(M)$, then $d\omega(X) = X(\omega)$,

(ii) $d \circ d = 0$,

(iii) If $\omega \in \Omega^k(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

Definition 1.4.8. We call a k-form *closed* if $d\omega = 0$, and well call it *exact* if there is a $(k-1)$-form η such that $d\eta = \omega$. Then we define the sets

$$\begin{aligned} Z^k(M) &= \ker(d : \Omega^k(M) \to \Omega^{k+1}(M)) && \text{(closed } k\text{-forms)}, \\ B^k(M) &= \operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M)) && \text{(exact } k\text{-forms)}. \end{aligned}$$

Note that an exact form is always closed: if $\omega = d\eta \in B^k(M)$, then $d\omega = d^2\eta = 0$, thus we find that $B^k(M) \subset Z^k(M)$.

Definition 1.4.9. We call the quotient

$$H^k_{dR}(M) = \frac{Z^k(M)}{B^k(M)},$$

the *k-th de Rham cohomology group* of a differentiable manifold M. We denote its dimension as a real vector space by

$$b_k(M) = \dim_{\mathbb{R}} H^k_{dR}(M),$$

the *k-th Betti number.*

Property 1.4.10. Let M be a differentiable manifold. Then

(i) $b_0(M)$ is the number of connected components of M,

(ii) M is connected if and only if $b_0(M) = 1$,

(iii) If $b_1(M) = 0$, then M is simply connected. The converse does not hold in general.

We now introduce some forms on a Riemannian manifold M which will be useful later in this thesis.

Definition 1.4.11. The *dual forms* to an orthonormal basis $\{e_1, \ldots, e_n\}$ are the 1-forms ω^i such that

$$\omega^i(e_j) = \delta_{ij}.$$

Definition 1.4.12. The connection forms of a Riemannian manifold M are the 1-forms ω_i^j such that

$$\omega_i^j(e_k) = \langle \nabla_{e_k} e_i, e_j \rangle,$$

and the *curvature forms* are the 2-forms Ω_i^j defined as

$$\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j.$$

Property 1.4.13. These forms have the following properties:

(i) $\omega_i^j = -\omega_j^i$,

(ii) $\omega_i^i = 0$,

(ii) $\Omega_j^i = -\Omega_i^j$,

(iv) $\Omega_i^i = 0$,

(v) $\Omega_j^i = \frac{1}{2}\sum_{k,l}\langle R(e_i, e_j)e_k, e_l\rangle \omega^k \wedge \omega^l$.

We will now apply these forms to submanifold theory. Let M^n be a Riemannian submanifold of a Riemannian manifold $\tilde{M}^m$. Let $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$ be a local orthonormal frame such that $\{e_1, \dots, e_n\}$ are tangent to M and $\{e_{n+1}, \dots, e_m\}$ are normal. Let $\{\omega^1, \dots, \omega^n\}$ be the dual frame of $\{e_1, \dots, e_n\}$.

We shall make use of the following convention on the ranges of indices unless mentioned otherwise:

$$1 \leq \alpha, \beta, \gamma, \dots \leq m; 1 \leq i, j, k, l \leq n; n+1 \leq r, s, t, \dots, \leq m.$$

We denote by $\tilde{\omega}_\beta^\alpha$ the connection forms of $\tilde{\nabla}$, the Levi-Civita connection of $\tilde{M}$, and we define the curvature 2-forms of $\tilde{M}$, restricted to M, as

$$\tilde{\Omega}_\beta^\alpha = \frac{1}{2}\sum_{k,l}\langle \tilde{R}(e_\alpha, e_\beta)e_k, e_l\rangle \omega^k \wedge \omega^l.$$

Theorem 1.4.14. *The* Cartan structure equations *are given by*

$$d\omega^i = -\sum_j \tilde{\omega}_j^i \wedge \omega^j, \tag{1.4.1}$$

$$d\tilde{\omega}_i^j = 2\sum_r \tilde{\omega}_r^i \wedge \tilde{\omega}_j^r + \sum_k \tilde{\omega}_k^i \wedge \tilde{\omega}_j^k + \tilde{\Omega}_i^j, \tag{1.4.2}$$

$$d\tilde{\omega}_i^r = \sum_j \tilde{\omega}_j^i \wedge \tilde{\omega}_r^j + \sum_s \tilde{\omega}_s^i \wedge \tilde{\omega}_r^s + \tilde{\Omega}_i^r, \tag{1.4.3}$$

$$d\tilde{\omega}_r^s = 2\sum_i \tilde{\omega}_i^r \wedge \tilde{\omega}_s^i + \sum_t \tilde{\omega}_t^r \wedge \tilde{\omega}_s^t + \tilde{\Omega}_r^s. \tag{1.4.4}$$

Chapter 2

Lagrangian submanifolds

This chapter is devoted to give properties that hold for Lagrangian submanifolds of complex space forms in general. We apply formulas of Gauss and Weingarten and the Gauss, Codazzi and Ricci equations to Lagrangian submanifolds, give an appropriate basis of the tangent space at a point which will prove very useful in proving certain theorems and propositions, and finally apply the Cartan structure equations to Lagrangian submanifolds. This chapter is based on [Che01; Che11; CO74a].

2.1 Properties

Let M^n be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4\tilde{c})$. Let us start by noting that the "power" of a Lagrangian submanifold comes from the property $J(T_pM) = T_p^{\perp}M$. This means that T_pM and $T_p^{\perp}M$ are in fact isomorphic, where the isomorphism is given by the almost complex structure J. This implies that any normal vector can be written as the image under J of a tangent vector, and vice versa.

Note that we will from now on assume that $n \geq 2$. If $n = 1$, then every 1-dimensional submanifold of a complex space form is Lagrangian.

As mentioned before, the main tools of submanifold theory are the formulas of Gauss and Weingarten and the fundamental equations of Gauss, Codazzi and Ricci. Therefore it is natural to see what happens to these formulas and equations for a Lagrangian submanifold.

Property 2.1.1. For a Lagrangian submanifold of a complex space form, we have that:

$$JA_{JY}X = h(X,Y), \tag{2.1.1}$$

$$\nabla^{\perp}_X JY = J\nabla_X Y, \tag{2.1.2}$$

$$A_{JX}Y = A_{JY}X. \tag{2.1.3}$$

Proof. Consider the Kähler condition $\tilde{\nabla} J \equiv 0$. We apply the formulas of Gauss and Weingarten:

$$0 = \tilde{\nabla}_X JY - J\tilde{\nabla}_X Y = -A_{JY}X + \nabla^{\perp}_X JY - J\nabla_X Y - Jh(X,Y).$$

The first two properties now follow by taking the tangent and normal components respectively. The third property follows from the first, since the second fundamental form h is symmetric in X and Y. □

The following $(3,0)$-tensor will be very important and will be one of the main tools to study Lagrangian submanifolds.

Definition 2.1.2. We define the *cubic form* C on T_pM as

$$C(X,Y,Z) = \langle h(X,Y), JZ\rangle.$$

Remark 2.1.3. If we are working with an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM, we will often write

$$C_{ijk} = C(e_i,e_j,e_k).$$

Property 2.1.4. The cubic form C is totally symmetric.

Proof. We can prove that C is symmetric in its first two components by using the symmetry of h:

$$C(X,Y,Z) = \langle h(X,Y), JZ\rangle = \langle h(Y,X), JZ\rangle = C(Y,X,Z).$$

Using (2.1.1) we find

$$C(X,Y,Z) = \langle A_{JX}Y, Z\rangle = \langle A_{JX}Z, Y\rangle = C(X,Z,Y),$$

so C is also symmetric in its last two components and is therefore totally symmetric. □

Let us now take a look at the curvature tensor $\tilde{R}$ of $\tilde{M}$.

Property 2.1.5. The curvature tensor $\tilde{R}$ of the complex space form $\tilde{M}(4\tilde{c})$ is

$$\tilde{R}(X,Y) = \tilde{c}\left((X\wedge Y) + (JX\wedge JY)\right). \tag{2.1.4}$$

In particular, we have that

$$\tilde{R}(X,Y)Z = \tilde{c}(X\wedge Y)Z, \tag{2.1.5}$$
$$\tilde{R}(X,Y)JZ = \tilde{c}(JX\wedge JY)JZ. \tag{2.1.6}$$

Proof. All of these equations follow from the fact that a Lagrangian submanifold is totally real. □

Next, we want to study equations of Gauss, Codazzi and Ricci for Lagrangian submanifolds. The equation of Gauss will give us information about the curvature tensor R of the Lagrangian submanifold:

Property 2.1.6. Let M be a Lagrangian submanifold of a complex space form $\tilde{M}(4\tilde{c})$. Then the curvature tensor R of M is

$$R(X,Y) = \tilde{c}(X\wedge Y) + [A_{JX}, A_{JY}]. \tag{2.1.7}$$

Proof. Consider the equation of Gauss (1.3.7). We have that

$$\langle h(Y,Z), h(X,W)\rangle = \langle JA_{JY}Z, JA_{JX}W\rangle = \langle A_{JY}Z, A_{JX}W\rangle = \langle A_{JX}A_{JY}Z, W\rangle, \quad (2.1.8)$$

and similarly we have

$$\langle h(X,Z), h(Y,W)\rangle = \langle A_{JY}A_{JX}Z, W\rangle. \quad (2.1.9)$$

So applying (2.1.4), (2.1.8) and (2.1.9) to the equation of Gauss, we end up with

$$\tilde{c}\langle (X \wedge Y)Z, W\rangle = \langle R(X,Y)Z, W\rangle - \langle A_{JX}A_{JY}Z, W\rangle + \langle A_{JY}A_{JX}Z, W\rangle,$$

which proves the property. □

Next, we consider the equation of Codazzi.

Property 2.1.7. For a Lagrangian submanifold M of a complex space form $\tilde{M}(4c)$, we have that $\bar{\nabla}h$ is totally symmetric.

Proof. Consider the equation of Codazzi (1.3.5). We know that $\tilde{R}(X,Y)Z$ has no normal components, so we end up with

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z),$$

which means that $(\bar{\nabla}_X h)(Y,Z)$ is symmetric in X and Y. But $(\bar{\nabla}_X h)(Y,Z)$ is always symmetric in Y and Z, so $\bar{\nabla}h$ is actually totally symmetric. □

Due to equations (2.1.2), (2.1.6) and (2.1.8), the equation of Ricci (1.3.6) for Lagrangian submanifolds is equivalent to the equation of Gauss.

2.2 Canonical basis

In this section, we introduce a *canonical basis* for the tangent space T_pM [Dil+12; Eji81; LV05; LW09; MU88], which will prove very useful in analysing the tangent space.

Let T be an n-dimension real vector space and C a totally symmetric $(3,0)$-tensor on T. Let us define

$$U = \{X \in T \mid \|X\| = 1\},$$

which is simply the unit hypersphere in T. We choose the vector

$$e_1 = \operatorname*{argmax}_{X\in U} C(X,X,X), \quad (2.2.1)$$

Since the $(2,0)$-tensor $C(e_1,X,Y)$ is symmetric in X and Y, it can be diagonalised. We show that e_1 is an eigenvector:

Lemma 2.2.1. *The vector e_1 defined in* (2.2.1) *is an eigenvector of the* $(2,0)$*-tensor* $C(e_1,X,Y)$.

Proof. We have to prove that $C(e_1, e_1, X)$ is zero for any vector X orthogonal to e_1. Take the curve

$$\gamma(t, X) : [0, 2\pi] \times \{X \in U \mid X \perp e_1\} \to U : (t, X) \mapsto \cos(t)e_1 + \sin(t)X.$$

Now, we have that

$$\begin{aligned} C(\gamma(t,X), \gamma(t,X), \gamma(t,X)) = &\cos^3(t)C(e_1,e_1,e_1) + 3\cos^2(t)\sin(t)C(e_1,e_1,X) \\ &+ 3\cos(t)\sin^2(t)C(e_1,X,X) + \sin^3(t)C(X,X,X). \end{aligned}$$

Differentiating this with respect to t gives

$$\begin{aligned} &\frac{d}{dt}C(\gamma(t,X), \gamma(t,X), \gamma(t,X)) \\ &\quad = -3\cos^2(t)\sin(t)C(e_1,e_1,e_1) + (3\cos^3(t) - 6\cos(t)\sin^2(t))C(e_1,e_1,X) \\ &\qquad + (6\cos^2(t)\sin(t) - 3\sin^3(t))C(e_1,X,X) + 3\cos(t)\sin^2(t)C(X,X,X). \end{aligned}$$

By definition of e_1, we know that $C(\gamma(t,X), \gamma(t,X), \gamma(t,X))$ reaches a maximum at $t = 0$. We evaluate the derivative in $t = 0$, so

$$0 = \frac{d}{dt}C(\gamma(t,X), \gamma(t,X), \gamma(t,X))\bigg|_{t=0} = 3C(e_1, e_1, X),$$

so we find that $C(e_1, e_1, X)$ must be zero and thus e_1 is an eigenvector. □

Definition 2.2.2. A *canonical basis* of T for a totally symmetric $(3,0)$-tensor C is an orthonormal basis $\{e_1, \ldots, e_n\}$ such that e_1 is defined as in (2.2.1) and all e_i are eigenvectors of the $(2,0)$-tensor $C(e_1, X, Y)$.

Now that we have a suitable basis of eigenvectors $\{e_1, \ldots, e_n\}$, we can study their eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$.

Property 2.2.3. A canonical basis $\{e_1, \ldots, e_n\}$ with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ satisfies the following:

(i) $C_{1ij} = \lambda_i \delta_{ij}$,

(ii) $\lambda_1 \geq 2\lambda_i$ for $2 \leq i \leq n$ and if $\lambda_1 = 2\lambda_i$, then $C_{iii} = 0$,

(iii) if $\lambda_1 = 0$, then $C \equiv 0$.

Proof. Item (i) follows directly from the fact that the e_i are all mutually orthogonal eigenvectors of the tensor $C(e_1, X, Y)$.

To prove item (ii), note that the second derivative to t of $C(\gamma(t,X), \gamma(t,X), \gamma(t,X))$ must be nonpositive since we have a maximum at $t = 0$. So we calculate this derivative

$$\begin{aligned} \frac{d^2}{dt^2}C(\gamma(t,X), \gamma(t,X), \gamma(t,X)) = &(-3\cos^3(t) + 6\cos(t)\sin^2(t))C(e_1,e_1,e_1) \\ &+ (-21\cos^2(t)sin(t) + 6\sin^3(t))C(e_1,e_1,X) \\ &+ (6\cos^3(t) - 21\cos(t))\sin^2(t)C(e_1,X,X) \\ &+ (6\cos^2(t)\sin(t) - 3\sin^3(t))C(X,X,X). \end{aligned}$$

Evaluating at $t = 0$ gives us

$$0 \geq \frac{d^2}{dt^2} C(\gamma(t,X),\gamma(t,X),\gamma(t,X))\Big|_{t=0} = -3C(e_1,e_1,e_1) + 6C(e_1,X,X),$$

so if we choose $X = e_i$ with $i \in \{2,\dots,n\}$ we indeed find that $\lambda_1 \geq 2\lambda_i$. Now suppose that equality is attained for some i: $\lambda_i = \lambda_1/2$. Then $\frac{d^2}{dt^2} C(\gamma(t,e_i),\gamma(t,e_i),\gamma(t,e_i))\Big|_{t=0} = 0$. But $C(\gamma(t,e_i),\gamma(t,e_i),\gamma(t,e_i))$ reaches a maximum at $t = 0$ so the third derivative must then be zero too. We calculate the third derivative:

$$\begin{aligned}\frac{d^3}{dt^3} C(\gamma(t,e_i),\gamma(t,e_i),\gamma(t,e_i)) = {} & (21\cos^2(t)\sin(t) - 6\sin^3(t))C_{111} \\ & + (-21\cos^3(t) + 60\cos(t)\sin^2(t))C_{11i} \\ & + (-60\cos^2(t)\sin(t) + 21\sin^3(t))C_{1ii} \\ & + (6\cos^3(t) - 21\cos(t)\sin^2(t))C_{iii}.\end{aligned}$$

So evaluating at $t = 0$, we get

$$0 = \frac{d^3}{dt^3} C(\gamma(t,e_i),\gamma(t,e_i),\gamma(t,e_i))\Big|_{t=0} = 6C_{iii},$$

so indeed C_{iii} is zero.

For item (iii), note that if $\lambda_1 = 0$, then $C(X,X,X) = 0$ for all $X \in T$. By linearity we obtain $C \equiv 0$. $\square$

Remark 2.2.4. If $\lambda_1 = 2\lambda_i$ for some $i \in 2,\dots,n$, we also know that the fourth derivative of $C(\gamma(t,e_i),\gamma(t,e_i),\gamma(t,e_i))$ must be nonpositive. However, calculating the fourth derivative gives us that $60\lambda_i \geq 21\lambda_1$, which is obviously the case.

Remark 2.2.5. In general there is no "unique" canonical basis. For example, if V is the m-dimensional eigenspace for an eigenvalue λ_V spanned by $\{e_i,\dots,e_{i+m-1}\}$, then for any $v \in V$ we find that $C(e_1,v,v) = \lambda_V$. So any orthonormal basis $\{f_1,\dots,f_m\}$ of V satisfies $C(e_1,f_i,f_j) = \lambda_V \delta_{ij}$, and we can replace $\{e_i,\dots,e_{i+m-1}\}$ by $\{f_1,\dots,f_m\}$ in the canonical basis without changing any of its properties.

Definition 2.2.6. Let M be a Lagrangian submanifold of a complex space form. A canonical basis for the vector space T_pM and the cubic form C will simply be called a *canonical basis* of T_pM.

Remark 2.2.7. Note that this basis forms a frame, i.e. the e_i all form local vector fields [Sza82]. If this were not the case, a vector field of the form $\nabla_{e_i} e_j$ would not be differentiable.

2.3 Cartan structural equations

We have introduced the Cartan structural equations before. In this section, we study how these equations behave when applied to a Lagrangian submanifold M of a complex space form. We shall denote by $\{e_1,\dots,e_n\}$ a basis for T_pM, thus $T_p^\perp M$ is spanned by $\{Je_1 = e_{n+1},\dots,Je_n = e_{2n}\}$. We find the following properties:

Property 2.3.1. Let M^n be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4\tilde{c})$. Denote by $\tilde{\omega}$ the connection forms of $\tilde{\nabla}$ and by $\tilde{\Omega}$ the curvature forms restricted to M. Then

(i) $\tilde{\omega}_i^j = \tilde{\omega}_{i+n}^{j+n}$,

(ii) $\tilde{\omega}_i^{j+n} = -\tilde{\omega}_{i+n}^j$,

(iii) $\tilde{\Omega}_j^i = \tilde{\Omega}_{j+n}^{i+n}$,

(iv) $\tilde{\Omega}_i^{i+n} = -\tilde{\Omega}_{j+n}^i$.

Proof. All properties follow from straightforward calculation:

$$\tilde{\omega}_i^j(e_\alpha) = \langle \tilde{\nabla}_{e_\alpha} e_i, e_j\rangle = \langle J\tilde{\nabla}_{e_\alpha} e_i, Je_j\rangle = \langle \tilde{\nabla}_{e_\alpha} Je_i, Je_j\rangle = \langle \tilde{\nabla}_{e_\alpha} e_{i+n}, e_{j+n}\rangle = \tilde{\omega}_{i+n}^{j+n}(e_\alpha),$$

$$\tilde{\omega}_i^{j+n}(e_\alpha) = \langle \tilde{\nabla}_{e_\alpha} e_i, e_{j+n}\rangle = -\langle J\tilde{\nabla}_{e_\alpha} Je_i, Je_j\rangle = -\langle \tilde{\nabla}_{e_\alpha} e_{i+n}, e_j\rangle = -\tilde{\omega}_{i+n}^j(e_\alpha),$$

$$\begin{aligned}\tilde{\Omega}_j^i &= \frac{1}{2}\sum_{k,l}\langle \tilde{R}(e_i,e_j)e_k,e_l\rangle \omega^k\wedge\omega^l = \frac{1}{2}\sum_{k,l}\langle \tilde{R}(Je_i,Je_j)e_k,e_l\rangle \omega^k\wedge\omega^l \\ &= \frac{1}{2}\sum_{k,l}\langle \tilde{R}(e_{i+n},e_{j+n})e_k,e_l\rangle \omega^k\wedge\omega^l = \tilde{\Omega}_{j+n}^{i+n},\end{aligned}$$

$$\begin{aligned}\tilde{\Omega}_i^{i+n} &= \frac{1}{2}\sum_{k,l}\langle \tilde{R}(e_i,e_{j+n})e_k,e_l\rangle \omega^k\wedge\omega^l = \frac{1}{2}\sum_{k,l}\langle \tilde{R}(e_i,Je_j)e_k,e_l\rangle \omega^k\wedge\omega^l \\ &= -\frac{1}{2}\sum_{k,l}\langle \tilde{R}(Je_i,e_j)e_k,e_l\rangle \omega^k\wedge\omega^l = -\frac{1}{2}\sum_{k,l}\langle \tilde{R}(e_{i+n},e_j)e_k,e_l\rangle \omega^k\wedge\omega^l = -\tilde{\Omega}_{j+n}^i,\end{aligned}$$

so all properties check out. ☐

Using these symmetries, we may rewrite the Cartan structure equations for Lagrangian submanifolds:

Theorem 2.3.2. *The Cartan structure equations for a Lagrangian submanifold of a complex space form are as follows:*

$$d\omega^i = -\sum_j \tilde{\omega}_j^i \wedge \omega^j, \tag{2.3.1}$$

$$d\tilde{\omega}_i^j = 2\sum_k \tilde{\omega}_{k+n}^i \wedge \tilde{\omega}_j^{k+n} + \sum_k \tilde{\omega}_k^i \wedge \tilde{\omega}_j^k + \tilde{\Omega}_i^j, \tag{2.3.2}$$

$$d\tilde{\omega}_i^{j+n} = \sum_k \tilde{\omega}_k^i \wedge \tilde{\omega}_{j+n}^k + \sum_k \tilde{\omega}_k^j \wedge \tilde{\omega}_{i+n}^k + \tilde{\Omega}_i^{j+n}, \tag{2.3.3}$$

Proof. These equations follow from writing every normal index r, s, t as a tangent index plus n in the Cartan structure equations, and applying the above properties. ☐

Remark 2.3.3. Since $\tilde{\omega}_i^j = \tilde{\omega}_{i+n}^{j+n}$, the Lagrangian equivalent of (1.4.4) is already given by (2.3.2).

Part I

Parallelity conditions on Lagrangian submanifolds

Chapter 3

Parallelity conditions

Up till now, there does not exist a complete classification of Lagrangian submanifolds of complex space forms. Therefore it seems useful to study and attempt to classify Lagrangian submanifolds with additional constraints. We will study constraints related to the mean curvature, the second fundamental form and the cubic form.

3.1 Notions of parallelity

In this section we introduce different notions of parallelity for a tensor T. Much of this section is based on [Dil+13].

One of the ways we can put a constraint on a Riemannian manifold M is to have a condition on a certain tensor $T(U_1, \dots, U_n)$ on T_pM. The most symmetric condition would be to demand the tensor vanishes, however, in many cases this is too strict. A weaker condition is the following:

Definition 3.1.1. Let T be a $(n,0)$- or $(n,1)$-tensor on a Riemannian manifold M. We call T *parallel* when $\nabla T \equiv 0$.

We could produce weaker conditions by taking higher-order derivatives, i.e. $\nabla^k T \equiv 0$ for some $k \in \mathbb{N}$. However, a different option exists. We could make the curvature tensor R act as a differentiation on a tensor, similar to how we defined differentiation by the Levi-Civita connection:

Definition 3.1.2. Consider a Riemannian manifold M. Then:

(i) if $T(U_1, \dots, U_n)$ is a $(n,0)$-tensor on M, we define:

$$\begin{aligned} R(X,Y) \cdot T(U_1, \dots, U_n) = \; & -T(R(X,Y)U_1, U_2, \dots, U_n) \\ & - \cdots - T(U_1, \dots, U_{n-1}, R(X,Y)U_n), \end{aligned}$$

(ii) if $T(U_1, \dots, U_n)$ is a $(n,1)$-tensor on M, we define:

$$\begin{aligned} R(X,Y) \cdot T(U_1, \dots, U_n) = \; & R(X,Y)T(U_1, \dots, U_n) - T(R(X,Y)U_1, U_2, \dots, U_n) \\ & - \cdots - T(U_1, \dots, U_{n-1}, R(X,Y)U_n). \end{aligned}$$

Definition 3.1.3. Let T be a $(n,0)$- or $(n,1)$-tensor on a Riemannian manifold M. We call T *semi-parallel* when $R \cdot T \equiv 0$.

This is a strictly weaker condition than $\nabla^2 T \equiv 0$, as implied by the following theorem:

Theorem 3.1.4 (Ricci identity). *Let T be a $(n,1)$-tensor. Then we have that*

$$R(X,Y) \cdot T(U_1, \dots, U_n) = (\nabla^2_{X,Y} T)(U_1, \dots, U_n) - (\nabla^2_{Y,X} T)(U_1, \dots, U_n).$$

Proof. Simply by expanding, we have

$$\begin{aligned}
(\nabla^2_{X,Y} T)(U_1, \dots, U_n) &= \nabla_X (\nabla_Y T)(U_1, \dots, U_n) - (\nabla_{\nabla_X Y} T)(U_1, \dots, U_n) \\
&\quad - \sum_{i=1}^n (\nabla_Y T)(U_1, \dots, \nabla_X U_i, \dots, U_n) \\
&= \nabla_X \nabla_Y T(U_1, \dots, U_n) - \sum_{i=1}^n \nabla_X T(U_1, \dots, \nabla_Y U_i, \dots, U_n) \\
&\quad - \nabla_{\nabla_X Y} T(U_1, \dots, U_n) + \sum_{i=1}^n T(U_1, \dots, \nabla_{\nabla_X Y} U_i, \dots, U_n) \\
&\quad - \sum_{i=1}^n \nabla_Y T(U_1, \dots, \nabla_X U_i, \dots, U_n) \\
&\quad + \sum_{i=1}^n T(U_1, \dots, \nabla_Y \nabla_X U_i, \dots, U_n) \\
&\quad + \sum_{i \neq j} T(U_1, \dots, \nabla_X U_i, \dots, \nabla_Y U_j, \dots, U_n),
\end{aligned}$$

and similarly we have

$$\begin{aligned}
(\nabla^2_{Y,X} T)(U_1, \dots, U_n) &= \nabla_Y \nabla_X T(U_1, \dots, U_n) - \sum_{i=1}^n \nabla_Y T(U_1, \dots, \nabla_X U_i, \dots, U_n) \\
&\quad - \nabla_{\nabla_Y X} T(U_1, \dots, U_n) + \sum_{i=1}^n T(U_1, \dots, \nabla_{\nabla_Y X} U_i, \dots, U_n) \\
&\quad - \sum_{i=1}^n \nabla_X T(U_1, \dots, \nabla_Y U_i, \dots, U_n) \\
&\quad + \sum_{i=1}^n T(U_1, \dots, \nabla_X \nabla_Y U_i, \dots, U_n) \\
&\quad + \sum_{i \neq j} T(U_1, \dots, \nabla_Y U_i, \dots, \nabla_X U_j, \dots, U_n).
\end{aligned}$$

Taking the difference between these two derivatives of T gives

$$\begin{aligned}
&(\nabla^2_{X,Y}T)(U_1,\dots,U_n) - (\nabla^2_{Y,X}T)(U_1,\dots,U_n)\\
&= \nabla_X\nabla_Y T(U_1,\dots,U_n) - \nabla_Y\nabla_X T(U_1,\dots,U_n)\\
&\quad - \nabla_{\nabla_X Y}T(U_1,\dots,U_n) + \nabla_{\nabla_Y X}T(U_1,\dots,U_n)\\
&\quad - \sum_{i=1}^n T(U_1,\dots,\nabla_X\nabla_Y U_i,\dots,U_n) + \sum_{i=1}^n T(U_1,\dots,\nabla_Y\nabla_X U_i,\dots,U_n)\\
&\quad - \sum_{i=1}^n T(U_1,\dots,\nabla_{\nabla_Y X} U_i,\dots,U_n) + \sum_{i=1}^n T(U_1,\dots,\nabla_{\nabla_X Y} U_i,\dots,U_n)\\
&= R(X,Y)T(U_1,\dots,U_n) - \sum_{i=1}^n T(U_1,\dots,R(X,Y)U_i,\dots,U_n)\\
&= R(X,Y)\cdot T(U_1,\dots,U_n),
\end{aligned}$$

which proves the Ricci identity. □

Similarly, for $(n,0)$-tensors we have the following:

Theorem 3.1.5 (Ricci Identity)**.** *Let T be a $(n,0)$-tensor. Then we have that*

$$R(X,Y)\cdot T(U_1,\dots,U_n) = (\nabla^2_{X,Y}T)(U_1,\dots,U_n) - (\nabla^2_{Y,X}T)(U_1,\dots,U_n).$$

Proof. The proof is identical to the previous Ricci identity, by replacing derivatives of the form $\nabla_X T(U_1,\dots,U_n)$ by the directional derivatives $X(T(U_1,\dots,U_n))$. □

We can produce an even weaker condition by using the wedge operator $\wedge$, which behaves like a curvature tensor.

Definition 3.1.6. Consider a Riemannian manifold M. Then:

(i) if $T(U_1,\dots,U_n)$ is a $(n,0)$-tensor on M, we define:

$$\begin{aligned}(X\wedge Y)\cdot T(U_1,\dots,U_n) = \;& -T((X\wedge Y)U_1,U_2,\dots,U_n)\\ & -\dots - T(U_1,\dots,U_{n-1},(X\wedge Y)U_n),\end{aligned}$$

(ii) if $T(U_1,\dots,U_n)$ is a $(n,1)$-tensor on M, we define:

$$\begin{aligned}(X\wedge Y)\cdot T(U_1,\dots,U_n) = \;& (X\wedge Y)T(U_1,\dots,U_n) - T((X\wedge Y)U_1,U_2,\dots,U_n)\\ & -\dots - T(U_1,\dots,U_{n-1},(X\wedge Y)U_n).\end{aligned}$$

We now define a new operator $R-\phi\wedge$ as

$$(R-\phi\wedge)(X,Y)\cdot T(U_1,\dots,U_n) = R(X,Y)\cdot T(U_1,\dots,U_n) - \phi(X\wedge Y)\cdot T(U_1,\dots,U_n),$$

where $\phi\in\mathcal{F}(M)$.

Definition 3.1.7. Let T be a $(n,0)$- or $(n,1)$-tensor on a Riemannian manifold M. We call T *pseudo-parallel* when $(R-\phi\wedge)\cdot T\equiv 0$, for any $\phi\in\mathcal{F}(M)$.

Note that for this condition to be useful, we will often require that the function ϕ is unique in a certain sense. This will become clear later in this thesis.

All of these notions of symmetry have been applied in an intrinsic context, by choosing $T = R$. The Riemannian manifolds with most symmetry are the flat ones, i.e. $R \equiv 0$. In the 1920's Cartan introduced in [Car26] the notion of *locally symmetric* spaces, being the Riemannian manifolds with $\nabla R = 0$, and later classified these with help of an older paper of his [Car14]. Later, he generalised these to the *semi-symmetric* manifolds having $R \cdot R \equiv 0$ [Car46], which were classified by Szabó [Sza82; Sza85]. Deszcz defined the *pseudo-symmetric* manifolds as those satisfying $(R - \phi\wedge) \cdot R \equiv 0$ [DG87].

It was proven in [DDV97] that for a Kähler manifold of (real) dimension $n \geq 4$, the only pseudo-symmetric Kähler manifolds are the semi-symmetric ones. For dimension $n = 2$, an example of a non-semi-symmetric pseudo-symmetric Kähler manifold was given in [Ols03].

Remark 3.1.8. In the definition of semi-symmetric manifolds, the second curvature tensor in $R \cdot R$ may be ambiguous: one can interpret it as a $(3,1)$-tensor $R(X,Y)Z$, but often it is also interpreted as a $(4,0)$-tensor $R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$. This confusion is possible in general: an $(n,1)$-tensor $T(U_1,\ldots,U_n)$ can be interpreted as a $(n+1,0)$-tensor $T'(U_1,\ldots,U_n,U_{n+1}) = \langle T(U_1,\ldots,U_n), U_{n+1}\rangle$ by combining it with the metric. However, the next proposition shows this doesn't matter.

Proposition 3.1.9. *Let T be a $(n,1)$-tensor, and define T' as*

$$T'(U_1,\ldots,U_n,U_{n+1}) = \langle T(U_1,\ldots,U_n), U_{n+1}\rangle,$$

so it is a $(n+1,0)$-tensor. Then

(i) $\langle R(X,Y)\cdot T(U_1,\ldots,U_n), U_{n+1}\rangle = R(X,Y)\cdot T'(U_1,\ldots,U_n,U_{n+1})$,

(ii) $\langle (X\wedge Y)\cdot T(U_1,\ldots,U_n), U_{n+1}\rangle = (X\wedge Y)\cdot T'(U_1,\ldots,U_n,U_{n+1})$.

Proof. Item (i) follows from a direct computation:

$$\begin{aligned}
R(X,Y)\cdot T'(U_1,\ldots,U_n,U_{n+1}) &= -\sum_{i=1}^{n+1} T'(U_1,\ldots,R(X,Y)U_i,\ldots,U_{n+1}) \\
&= -\sum_{i=1}^{n}\langle T(U_1,\ldots,R(X,Y)U_i,\ldots,U_n), U_{n+1}\rangle \\
&\quad - \langle T(U_1,\ldots,U_n), R(X,Y)U_{n+1}\rangle \\
&= -\sum_{i=1}^{n}\langle T(U_1,\ldots,R(X,Y)U_i,\ldots,U_n), U_{n+1}\rangle \\
&\quad + \langle R(X,Y)T(U_1,\ldots,U_n), U_{n+1}\rangle \\
&= \langle R(X,Y)\cdot T(U_1,\ldots,U_n), U_{n+1}\rangle.
\end{aligned}$$

Because the wedge has the same symmetries as a curvature tensor, item (ii) is proven in the exact same way as item (i). □

In submanifold theory we look at extrinsic properties, e.g. the mean curvature H, the second fundamental form h or the cubic form C. If we are considering a tensor that returns a tangent vector or a scalar (e.g. the cubic form), we can naturally apply all the previous notions of parallelity. However, in the case where we have a normal vector (e.g. the second fundamental form or mean curvature) we will not use ∇ and R to differentiate.

Instead, we will use the Van der Waerden-Bortolotti connection $\bar{\nabla}$ and its associated curvature tensor $\bar{R} = R \oplus R^{\perp}$. As the wedge operator $\wedge$ does not have a "normal counterpart" like $\nabla^{\perp}$ for ∇ and $R^{\perp}$ for R, we cannot do something similar for it.

Definition 3.1.10. Let $N(U_1, \ldots, U_n)$ be a $(n,1)$-tensor returning a normal vector. We define:

$$\begin{aligned}
(\bar{\nabla}_X N)(U_1, \ldots, U_n) &= \nabla^{\perp}_X N(U_1, \ldots, U_n) - N(\nabla_X U_1, U_2, \ldots, U_n) \\
&\quad - \cdots - N(U_1, \ldots, U_{n-1}, \nabla_X U_n), \\
\bar{R}(X,Y) \cdot N(U_1, \ldots, U_n) &= R^{\perp}(X,Y) N(U_1, \ldots, U_n) - N(R(X,Y)U_1, U_2, \ldots, U_n), \\
&\quad - \cdots - N(U_1, \ldots, U_{n-1}, R(X,Y)U_n), \\
(X \wedge Y) \cdot N(U_1, \ldots, U_n) &= -N((X \wedge Y)U_1, U_2, \ldots, U_n) \\
&\quad - \cdots - N(U_1, \ldots, U_{n-1}, (X \wedge Y)U_n).
\end{aligned}$$

We also name these conditions like before.

Definition 3.1.11. Let N be a $(n,1)$-tensor on a Riemannian submanifold M. Then we say that N is *parallel* if $\bar{\nabla} N \equiv 0$, that it is *semi-parallel* if $\bar{R} \cdot N \equiv 0$ and finally that it is *pseudo-parallel* if $(\bar{R} - \phi\wedge) \cdot N \equiv 0$ for any $\phi \in \mathcal{F}(M)$.

Similar properties as before hold. First, we have the Ricci identity for a $(n,1)$-tensor returning a normal vector:

Theorem 3.1.12 (Ricci Identity)**.** *Let N be a $(n,1)$-tensor returning a normal vector field. Then we have that [MU88]:*

$$\bar{R}(X,Y) \cdot N(U_1, \ldots, U_n) = (\bar{\nabla}^2_{X,Y} N)(U_1, \ldots, U_n) - (\bar{\nabla}^2_{Y,X} N)(U_1, \ldots, U_n).$$

Proof. Again, the proof is identical to the first Ricci identity, by replacing the derivatives of the form $\nabla_X T(U_1, \ldots, U_n)$ by $\nabla^{\perp}_X N(U_1, \ldots, U_n)$. □

Now, let us restrict ourselves to Lagrangian submanifolds. We have that J "commutes" with the Levi-Civita connection ∇ and the curvature tensor R when these are used for taking derivatives.

Theorem 3.1.13. *Suppose we have a $(n,1)$-tensor N returning a normal vector, then JN is a $(n,1)$-tensor returning a tangent vector field. We have that*

(i) for any $k \in \mathbb{N}$, $J(\bar{\nabla}^k_{X_1,\ldots,X_k} N)(U_1, \ldots, U_n) = (\nabla^k_{X_1,\ldots,X_k} JN)(U_1, \ldots, U_n)$,

(ii) $J(\bar{R}(X,Y) \cdot N)(U_1, \ldots, U_n) = R(X,Y) \cdot JN(U_1, \ldots, U_n)$.

Proof. The proof is quite straightforward using the definitions of the differentiations. For item (i), let us work by induction. First consider the case $k = 1$.

$$\begin{aligned} J(\bar{\nabla}_X N)(U_1, \ldots, U_n) &= J\nabla^\perp_X N(U_1, \ldots, U_n) - J\sum_{i=1}^n N(U_1, \ldots, \nabla_X U_i, \ldots, U_n) \\ &= \nabla_X JN(U_1, \ldots, U_n) - \sum_{i=1}^n JN(U_1, \ldots, \nabla_X U_i, \ldots, U_n) \\ &= (\nabla_X JN)(U_1, \ldots, U_n). \end{aligned}$$

Now suppose that

$$J(\bar{\nabla}^{k-1}_{X_1,\ldots,X_{k-1}} N)(U_1, \ldots, U_n) = (\nabla^{k-1}_{X_1,\ldots,X_{k-1}} JN)(U_1, \ldots, U_n),$$

then we find

$$\begin{aligned} &J(\bar{\nabla}^k_{X_1,\ldots,X_k} N)(U_1, \ldots, U_n) \\ &= J\nabla^\perp_{X_1}(\bar{\nabla}^{k-1}_{X_2,\ldots,X_k} N)(U_1, \ldots, U_n) - J\sum_{i=1}^n (\bar{\nabla}^{k-1}_{X_2,\ldots,X_k} N)(U_1, \ldots, \nabla_{X_1} U_i, \ldots, U_n) \\ &= \nabla_{X_1} J(\bar{\nabla}^{k-1}_{X_2,\ldots,X_k} N)(U_1, \ldots, U_n) - \sum_{i=1}^n J(\bar{\nabla}^{k-1}_{X_2,\ldots,X_k} N)(U_1, \ldots, \nabla_{X_1} U_i, \ldots, U_n) \\ &= \nabla_{X_1}(\nabla^{k-1}_{X_2,\ldots,X_k} JN)(U_1, \ldots, U_n) - \sum_{i=1}^n (\nabla^{k-1}_{X_2,\ldots,X_k} JN)(U_1, \ldots, \nabla_{X_1} U_i, \ldots, U_n) \\ &= (\nabla^k_{X_1,\ldots,X_k} JN)(U_1, \ldots, U_n). \end{aligned}$$

Item (ii) follows from taking $k = 2$ in item (i) and applying the Ricci identities. □

We also find that $\bar{\nabla}$ and $\bar{R}$ behave well with respect to the metric.

Theorem 3.1.14. *Suppose we have a $(n, 1)$-tensor N returning a normal vector. Define the $(n + 1, 0)$-tensor N' as*

$$N'(U_1, \ldots, U_n, U_{n+1}) = \langle N(U_1, \ldots, U_n), JU_{n+1}\rangle.$$

Then we have that

(i) for any $k \in \mathbb{N}$, $\langle(\bar{\nabla}^k_{X_1,\ldots,X_k} N)(U_1, \ldots, U_n), JU_{n+1}\rangle = (\nabla^k_{X_1,\ldots,X_k} N')(U_1, \ldots, U_n, U_{n+1})$,

(ii) $\langle\bar{R}(X_1, X_2) \cdot N(U_1, \ldots, U_n), JU_{n+1}\rangle = R(X_1, X_2) \cdot N'(U_1, \ldots, U_n, U_{n+1})$.

Proof. For item (i), let us work by induction. First consider the case $k = 1$:

$$\begin{aligned}
\langle(\bar{\nabla}_X N)(U_1, \dots, U_n), JU_{n+1}\rangle &= \langle\nabla^\perp_X N(U_1, \dots, U_n), JU_{n+1}\rangle \\
&\quad - \sum_{j=1}^{n} \langle N(U_1, \dots, \nabla_X U_i, \dots, U_n), JU_{n+1}\rangle \\
&= X\langle N(U_1, \dots, U_n), JU_{n+1}\rangle - \langle N(U_1, \dots, U_n), \nabla^\perp_X JU_{n+1}\rangle \\
&\quad - \sum_{j=1}^{n} N'(U_1, \dots, \nabla_X U_i, \dots, U_n, U_{n+1}) \\
&= X(N'(U_1, \dots, U_n, U_{n+1})) - N'(U_1, \dots, U_n, \nabla_X U_{n+1}) \\
&\quad - \sum_{j=1}^{n} N'(U_1, \dots, \nabla_X U_i, \dots, U_n, U_{n+1}) \\
&= (\nabla_X N')(U_1, \dots, U_n, U_{n+1}).
\end{aligned}$$

Now assume that

$$\langle(\bar{\nabla}^{k-1}_{X_1,\dots,X_{k-1}} N)(U_1, \dots, U_n), JU_{n+1}\rangle = (\nabla^{k-1}_{X_1,\dots,X_{k-1}} N')(U_1, \dots, U_n, U_{n+1}).$$

Then we find that

$$\begin{aligned}
\langle(\bar{\nabla}^{k}_{X_1,\dots,X_k} N)(U_1, \dots, U_n), JU_{n+1}\rangle &= \langle\nabla^\perp_{X_1}(\bar{\nabla}^{k-1}_{X_2,\dots,X_k} N)(U_1, \dots, U_n), JU_{n+1}\rangle \\
&\quad - \sum_{j=1}^{n} \langle(\bar{\nabla}^{k-1}_{X_2,\dots,X_k} N)(U_1, \dots, \nabla_{X_1} U_i, \dots, U_n), JU_{n+1}\rangle \\
&= X_1\langle(\bar{\nabla}^{k-1}_{X_2,\dots,X_k} N)(U_1, \dots, U_n), JU_{n+1}\rangle \\
&\quad - \langle(\bar{\nabla}^{k-1}_{X_2,\dots,X_k} N)(U_1, \dots, U_n), \nabla^\perp_{X_1} JU_{n+1}\rangle \\
&\quad - \sum_{j=1}^{n} (\nabla^{k-1}_{X_2,\dots,X_k} N')(U_1, \dots, \nabla_{X_1} U_i, \dots, U_n, U_{n+1}) \\
&= X_1((\nabla^{k-1}_{X_2,\dots,X_k} N')(U_1, \dots, U_n, U_{n+1})) \\
&\quad - (\nabla^{k-1}_{X_2,\dots,X_k} N')(U_1, \dots, U_n, \nabla_{X_1} U_{n+1}) \\
&\quad - \sum_{j=1}^{n} (\nabla^{k-1}_{X_2,\dots,X_k} N')(U_1, \dots, \nabla_{X_1} U_i, \dots, U_n, U_{n+1}) \\
&= (\nabla^{k}_{X_1,\dots,X_k} N')(U_1, \dots, U_n, U_{n+1}).
\end{aligned}$$

Item (ii) follows from taking $k = 2$ in item (i) and applying the Ricci identities. □

As mentioned before, the wedge operator $\wedge$ does not have a "normal" counterpart. Therefore the differentiation by the wedge does have these nice symmetries. However one may wonder if it is possible to define, at least in the Lagrangian setting, a normal counterpart $\wedge^\perp$?

The answer is yes. We can do this by imposing that the wedge and its normal counterpart must behave similar to R and $R^\perp$, which makes sense since the wedge operator behaves like a curvature operator. Since $R^\perp J = JR$ or thus $R^\perp = -JRJ$, we can define the normal wedge as follows:

Definition 3.1.15. The *normal wedge* $\wedge^{\perp}$ is defined as

$$X \wedge^{\perp} Y = -J(X \wedge Y)J = JX \wedge JY,$$

and we define the *Lagrangian wedge* as

$$\bar{\wedge} = \wedge \oplus \wedge^{\perp}.$$

Since the Lagrangian wedge has the exact same symmetries as the Van der Waerden-Bortolotti curvature, we find the following:

Theorem 3.1.16. *Suppose we have a $(n,1)$-tensor N returning a normal vector field, then JN is a $(n,1)$-tensor returning a tangent vector field. Moreover, define the $(n+1,0)$-tensor N' as*

$$N'(Y_1,\ldots,Y_n,Y_{n+1}) = \langle N(Y_1,\ldots,Y_n), JY_{n+1}\rangle.$$

We have that

(i) $J((X_1 \bar{\wedge} X_2)\cdot N)(Y_1,\ldots,Y_n)) = (X_1 \wedge X_2)\cdot JN(Y_1,\ldots,Y_n)$,

(ii) $\langle (X_1 \bar{\wedge} X_2)\cdot N(Y_1,\ldots,Y_n), JY_{n+1}\rangle = (X_1 \wedge X_2)\cdot N'(Y_1,\ldots,Y_n,Y_{n+1})$.

Definition 3.1.17. Let N be a $(n,1)$-tensor on a Riemannian submanifold M. Then we say that N is *Lagrangian pseudo-parallel* if $(\bar{R} - \phi\bar{\wedge})\cdot N \equiv 0$ for any $\phi \in \mathcal{F}(M)$.

This is again a weaker condition than semi-parallelity, but in general there is no relation between pseudo-parallelity and Lagrangian pseudo-parallelity. In a certain sense Lagrangian pseudo-parallelity is a more “logical” generalisation of semi-parallelity than pseudo-parallelity.

One the one hand, we see that the Lagrangian wedge retains the symmetries that the Van der Waerden-Bortolotti curvature has when used as a differentiation. On the other hand, it seems unnatural that when M has constant sectional curvature, and thus the curvature tensor R is a multiple of the wedge operator, R and $\wedge$ act differently when used to derive a tensor. For the Lagrangian wedge this is not the case. Moreover, we obtain the nice identity that if M has constant sectional curvature c, $R^{\perp} = c\wedge^{\perp}$.

3.2 Mean curvature

Often studied is the mean curvature H, which is the normalized trace of the second fundamental form h. Most commonly known are the *minimal* submanifolds, who have $H = 0$. By differentiating H in the ways mentioned in the previous section, we can produce weaker conditions. We shall give the following definitions:

Definition 3.2.1. We define the following conditions related to the mean curvature H:

Name	**Condition**
Totally geodesic	$h \equiv 0$
Minimal	$H = 0$
H-parallel / Parallel mean curvature	$\bar{\nabla} H = \nabla^{\perp} H \equiv 0$
H-semi-parallel	$\bar{R}\cdot H = R^{\perp} H \equiv 0$
H-pseudo-parallel	$(\bar{R} - \phi\bar{\wedge})\cdot H = (R^{\perp} - \phi\wedge^{\perp})H \equiv 0$

Proposition 3.2.2. *Every condition in the above table implies the next:*

$$h \equiv 0 \implies H = 0 \implies \nabla^\perp H \equiv 0 \implies R^\perp H \equiv 0 \implies (R^\perp - \phi\wedge^\perp)H \equiv 0.$$

Remark 3.2.3. The condition of "$R^\perp H = 0$" was first introduced by Deprez in [Dep85] (with no specific name given), and was first named "H-parallel" in a preprint of [Dil+13]. However, this was rather unfortunate, as "parallel mean curvature" and "H-parallel" would then be two different notions which could easily be confused with each other. In the published version of [Dil+13], the condition of $R^\perp H = 0$ was renamed to "H-semi-parallel". We will use "H-parallel" to denote $\nabla^\perp H \equiv 0$ and "H-semi-parallel" to denote $R^\perp H \equiv 0$.

Remark 3.2.4. We have defined the notion of H-pseudo-parallel using the Lagrangian wedge instead of the standard one. The reason for this is that $\wedge \cdot H$ is trivial. This would have made the condition of H-pseudo-parallel equivalent to H-semi-parallel, which is not very useful. The definition given here is in general not equivalent to H-semi-parallel.

The simplest kind of Lagrangian submanifolds are the totally geodesic ones, i.e. the Lagrangian submanifolds with $h \equiv 0$.

Theorem 3.2.5. *Let M^n be a Lagrangian submanifold. Then M is totally geodesic if and only if M has constant sectional curvature $\tilde{c}$ and M is minimal.*

Proof. If M is totally geodesic, the curvature tensor (2.1.7) becomes

$$R(X, Y) = \tilde{c}(X \wedge Y),$$

so M is a real space form of constant curvature $\tilde{c}$. Because H is the normalised trace of h, clearly M is minimal.

Conversely, assume M has constant sectional curvature $\tilde{c}$ and $H = 0$. Then we know that

$$\langle R(X, Y)Z, W\rangle = \langle \tilde{c}(X \wedge Y)Z, W\rangle = \langle \tilde{R}(X, Y)Z, W\rangle,$$

so the equation of Gauss (1.3.7) becomes

$$\langle h(Y, Z), h(X, W)\rangle - \langle h(X, Z), h(Y, W)\rangle = 0.$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for T_pM and choose $X = Z = e_i$ and $Y = W = e_j$, and take the sum over all $1 \leq i, j \leq n$:

$$\begin{aligned}
0 &= \sum_{i,j} \left(\langle h(e_i, e_i), h(e_j, e_j)\rangle - \langle h(e_i, e_j), h(e_i, e_j)\rangle\right) \\
&= \left\langle \sum_{i=1}^n h(e_i, e_i), \sum_{j=1}^n h(e_j, e_j) \right\rangle - \sum_{i,j} \langle h(e_i, e_j), h(e_i, e_j)\rangle \\
&= n^2 \langle H, H\rangle - \sum_{i,j=1}^n \|h(e_i, e_j)\|^2 \\
&= -\sum_{i,j} \|h(e_i, e_j)\|^2.
\end{aligned}$$

Since the norm is a positive function, we get $h(e_i, e_j) = 0$ for every $1 \leq i, j \leq n$ and therefore M is totally geodesic. $\square$

But what for minimal Lagrangian submanifolds of constant curvature c in general? We first introduce the following lemma:

Lemma 3.2.6. *Suppose there exists a positive constant α such that*

$$\begin{aligned}&\alpha(\langle X,Z\rangle\langle Y,W\rangle-\langle X,W\rangle\langle Y,Z\rangle)\\&+\textstyle\sum_{i=1}^n C(X,Z,e_i)C(Y,W,e_i)-\sum_{i=1}^n C(X,W,e_i)C(Y,Z,e_i)=0,\end{aligned} \tag{3.2.1}$$

and

$$\sum_{i=1}^n C(X,e_i,e_i)=0, \tag{3.2.2}$$

where $\{e_1,\ldots,e_n\}$ is an orthonormal basis of $\mathbb{R}^n$. We take the basis $\{f_1,\ldots,f_n\}$ where $f_i=\operatorname{argmax}_{X\in(U_pM)_i}C(X,X,X)$ with $(U_pM)_i=\{X\in U_pM\mid X\perp f_1,\ldots,f_{i-1}\}$. In particular, f_1 is the e_1-vector from a canonical basis.

Then C has the following expression with respect to the basis $\{f_1,\ldots,f_n\}$:

$$\begin{aligned}C_{iii}&=(n-i)\sqrt{\frac{\alpha}{n-i+1}\left(\sum_{z\in\mathbb{Z}_2^{i-1}}\prod_{a=1}^{i-1}\frac{1}{(n-i+1+a)^{z_a}}\right)},\\C_{ijj}&=-\sqrt{\frac{\alpha}{n-i+1}\left(\sum_{z\in\mathbb{Z}_2^{i-1}}\prod_{a=1}^{i-1}\frac{1}{(n-i+1+a)^{z_a}}\right)},\\C_{ijk}&=0,\end{aligned}$$

where $1\le i,j,k\le n$ and $i<j<k$ [Eji82].

Proof. Let $\{e_1,\ldots,e_n\}$ be a canonical basis. Choosing $X=Z=e_1$ and $Y=W=e_i$ in (3.2.1) where $i>1$ we find

$$\lambda_i^2-\lambda_1\lambda_i-\alpha=0,$$

which has 2 solutions, being

$$\lambda_i=\frac{1}{2}\lambda_1\pm\frac{1}{2}\sqrt{\lambda_1^2+4\alpha}.$$

However, only the choice for minus satisfies $\lambda_i\le\lambda_1/2$, which is required by the canonical basis. Choosing $X=e_1$ in (3.2.2) we have that

$$\lambda_1+\lambda_2+\ldots+\lambda_n=0.$$

Since there is only one possible value for $\lambda_2,\ldots,\lambda_n$, they are all equal and in particular we find

$$\begin{aligned}\lambda_1&=(n-1)\sqrt{\frac{\alpha}{n}},\\\lambda_2=\ldots=\lambda_n&=-\sqrt{\frac{\alpha}{n}}.\end{aligned}$$

We can now prove the lemma by induction. Let us first consider the case $n = 2$. Note that in this case we may assume $f_2 = e_2$. We then know that

$$C_{111} = \lambda_1 = \sqrt{\frac{\alpha}{2}},$$
$$C_{122} = \lambda_2 = -\sqrt{\frac{\alpha}{2}}.$$

By (3.2.2) and the properties of the canonical basis, we find

$$C_{222} = -C_{112} = 0.$$

Then all values for the case $n = 2$ check out. We now assume the form of the basis is true for dimension $\leq n-1$ and we consider the lemma for dimension n. Now, if $\{X', Y', Z', W'\}$ are spanned by $\{e_2, \ldots, e_n\}$, then

$$C(e_1, X', Y') = \lambda_1 \langle X', Y' \rangle,$$

and as a result we find that

$$\alpha\left(1 + \frac{1}{n}\right)(\langle X', Z'\rangle\langle Y', W'\rangle - \langle X', W'\rangle\langle Y', Z'\rangle)$$
$$+ \sum_{i=2}^{n} C(X', Z', e_i)C(Y', W', e_i) - \sum_{i=2}^{n} C(X', W', e_i)C(Y', Z', e_i) = 0,$$

and

$$\sum_{i=2}^{n} C(X', e_i, e_i) = 0.$$

Since $\alpha\left(1 + \frac{1}{n}\right)$ is still a positive constant, by the induction hypothesis the subspace spanned by $e_2, \ldots, e_n$ has the required form (after a transformation of basis to $f_2, \ldots, f_n$). Remains to verify the values containing at least one index 1. But $C_{111} = \lambda_1$ and C_{1ii} with $2 \leq n$ attain the correct values, since they equal the eigenvalues λ_1 and λ_i respectively. Finally, C_{11i} and C_{1ij} for $2 \leq i \neq j \leq n$ are all zero due to the properties of the canonical basis. This proves the lemma. □

Remark 3.2.7. The proof given above is much shorter than the original given by Ejiri in [Eji82]. In said article the property $\lambda_i \leq \lambda_1/2$ was not used, resulting in a long and complicated calculation to show that all eigenvalues λ_i $(i \geq 2)$ are equal.

Theorem 3.2.8. *A minimal Lagrangian submanifold of constant sectional curvature c is either flat or totally geodesic [Eji82].*

Proof. The equation of Gauss and minimality imply the previous lemma applies, with $\alpha = \tilde{c} - c$. If $\alpha = 0$, then by lemma 3.2.6, $C \equiv 0$ and M is totally geodesic. If $\alpha < 0$, then we could follow the steps of lemma 3.2.6 to find that $\lambda_1 \in \mathbb{C}\backslash\mathbb{R}$, which is impossible. So we assume $\alpha > 0$. We use the equation of Codazzi (1.3.5):

$$\nabla^{\perp}_{e_1} h(e_i, e_1) - h(\nabla_{e_1} e_i, e_1) - h(e_i, \nabla_{e_1} e_1) - \nabla^{\perp}_{e_i} h(e_1, e_1) + 2h(\nabla_{e_i} e_1, e_1) = 0.$$

We split up the Levi-Civita connection ∇ of M in its connection forms ω_i^j. We will do this term by term:

$$\begin{aligned}\nabla^{\perp}_{e_1} h(e_i, e_1) &= \nabla^{\perp}_{e_1} \sum_{j=1}^{n} \langle h(e_i, e_1), Je_j\rangle Je_j = \sum_{j=1}^{n} C(e_1, e_i, e_j) J\nabla_{e_1} e_j \\ &= -\sqrt{\frac{\tilde{c}-c}{n}} \sum_{j=1}^{n} \omega_i^j(e_1) Je_j,\end{aligned}$$

$$h(\nabla_{e_1} e_i, e_1) = h\left(\sum_{j=1}^{n} \omega_i^j(e_1) e_j, e_1\right) = \sum_{j=1}^{n} \omega_i^j(e_1) h\left(e_j, e_1\right),$$

$$h(e_i, \nabla_{e_1} e_1) = h\left(e_i, \sum_{j=1}^{n} \omega_1^j(e_1) e_j\right) = \sum_{j=1}^{n} \omega_1^j(e_1) h\left(e_i, e_j\right),$$

$$\begin{aligned}\nabla^{\perp}_{e_i} h(e_1, e_1) &= \nabla^{\perp}_{e_i} \sum_{j=1}^{n} \langle h(e_1, e_1), Je_j\rangle Je_j = \sum_{j=1}^{n} C(e_1, e_1, e_j) J\nabla_{e_i} e_j \\ &= (n-1)\sqrt{\frac{\tilde{c}-c}{n}} \sum_{j=1}^{n} \omega_1^j(e_i) Je_j,\end{aligned}$$

$$h(\nabla_{e_i} e_1, e_1) = h\left(\sum_{j=1}^{n} \omega_1^j(e_i) e_j, e_1\right) = \sum_{j=1}^{n} \omega_1^j(e_i) h\left(e_j, e_1\right),$$

so we end up with

$$\begin{aligned}&-\sqrt{\frac{\tilde{c}-c}{n}} \sum_{j=1}^{n} \omega_i^j(e_1) Je_j - \sum_{i=j}^{n} \omega_i^j(e_1) h\left(e_j, e_1\right) - \sum_{i=j}^{n} \omega_1^j(e_1) h\left(e_i, e_j\right) \\ &\quad - (n-1)\sqrt{\frac{\tilde{c}-c}{n}} \sum_{j=1}^{n} \omega_1^j(e_i) Je_j + 2\sum_{j=1}^{n} \omega_1^j(e_i) h\left(e_j, e_1\right) = 0.\end{aligned}$$

We take the inner product of this with Je_1, and using that ω is skew-symmetric in its indices, we find

$$\begin{aligned}0 = &-\sqrt{\frac{\tilde{c}-c}{n}} \omega_i^1(e_1) - \sum_{j=1}^{n} \omega_i^j(e_1) C\left(e_1, e_1, e_j\right) - \sum_{j=1}^{n} \omega_1^j(e_1) C\left(e_1, e_i, e_j\right) \\ &- (n-1)\sqrt{\frac{\tilde{c}-c}{n}} \omega_1^1(e_i) + 2\sum_{j=1}^{n} \omega_1^j(e_i) C\left(e_1, e_1, e_j\right) \\ = &\sqrt{\frac{\tilde{c}-c}{n}} \omega_1^i(e_1) + (n-1)\sqrt{\frac{\tilde{c}-c}{n}} \omega_1^i(e_1) + \sqrt{\frac{\tilde{c}-c}{n}} \omega_1^i(e_1) \\ = &(n+1)\sqrt{\frac{\tilde{c}-c}{n}} \omega_1^i(e_1).\end{aligned}$$

Since $\tilde{c} \neq c$, we find that $\omega_1^i(e_1) = 0$ for all $i \in \{1, \ldots, n\}$. Likewise, take the inner product with Je_k $(k \neq 1)$, using the previous result together with the properties of the connection forms:

$$\begin{aligned}
0 = & -\sqrt{\frac{\tilde{c}-c}{n}}\omega_i^k(e_1) - \sum_{j=1}^n \omega_i^j(e_1) C(e_1, e_k, e_j) - \sum_{j=1}^n \omega_1^j(e_1) C(e_i, e_k, e_j) \\
& -(n-1)\sqrt{\frac{\tilde{c}-c}{n}}\omega_1^k(e_i) + 2\sum_{j=1}^n \omega_1^j(e_i) C(e_1, e_k, e_j) = 0 \\
= & -\sqrt{\frac{\tilde{c}-c}{n}}\omega_i^k(e_1) + \sqrt{\frac{\tilde{c}-c}{n}}\omega_i^k(e_1) - (n-1)\sqrt{\frac{\tilde{c}-c}{n}}\omega_1^k(e_i) + 2\sqrt{\frac{\tilde{c}-c}{n}}\omega_1^k(e_i) \\
= & -(n+1)\sqrt{\frac{\tilde{c}-c}{n}}\omega_1^k(e_i).
\end{aligned}$$

So $\omega_1^k(e_i) = 0$ for all $i \in \{1, \ldots, n\}$, $k \in \{2, \ldots, n\}$. Combining this with the previous result, we obtain that $\omega_1^j(e_i) = 0$ for all $i, j \in \{1, \ldots, n\}$. Now we have that for any $X \in T_pM$:

$$\nabla_X e_1 = \sum_{i=1}^n \langle X, e_i \rangle \nabla_{e_i} e_1 = \sum_{i=1}^n \langle X, e_i \rangle \sum_{j=1}^n \omega_1^j(e_i) e_j = 0.$$

Using the fact that the sectional curvature is constant, we find

$$c = K(e_i, e_1) = \langle R(e_i, e_1)e_1, e_i \rangle = \langle \nabla_{e_i}\nabla_{e_1} e_1 - \nabla_{e_1}\nabla_{e_i} e_1 - \nabla_{[e_i, e_1]} e_1, e_i \rangle = 0,$$

thus M is flat. □

We may weaken the condition of M being minimal in theorem 3.2.8 to H-semi-parallel.

Corollary 3.2.9. *Lagrangian submanifolds of constant sectional curvature c are H-semi-parallel if and only if they are flat or totally geodesic.*

Proof. First suppose that M^n is a H-semi-parallel Lagrangian submanifold of constant sectional curvature c. Let X be a unit vector orthogonal to JH. We have that

$$0 = \langle R^\perp(JH, X)H, JX \rangle = -\langle H, JR(JH, X)X \rangle = -c\langle H, J(JH \wedge X)X \rangle = c\|H\|^2.$$

Thus have that M is flat or minimal, and by the previous theorem, it is flat or totally geodesic.

Conversely, if M is flat then $R^\perp \equiv 0$, and if it is totally geodesic then $H = 0$, both imply that $R^\perp H \equiv 0$ and thus M is H-semi-parallel. □

The function $\phi \in \mathcal{F}(M)$ in the definition of H-pseudo-parallelity is unique in some sense.

Theorem 3.2.10. *Let M be a Lagrangian submanifold that is H-pseudo-parallel for functions ϕ and psi. Then $\phi = \psi$ on the set $M \backslash \{p \in M \mid H_p \equiv 0\}$.*

Proof. If M is H-pseudo-parallel for both the functions ϕ and ψ, then we have that

$$(\phi - \psi)(JX \wedge JY)H = 0,$$

for all $X, Y \in T_pM$. Choose Y to be a unit vector such that JY is orthogonal to H and choose $X = JH$, then we find that

$$(\phi - \psi)\|H\|^2 = 0.$$

Now let $p \in M$ such that $\phi(p) \neq \psi(p)$, then $H(p) = 0$. Consequently,

$$\{p \in M \mid \phi(p) \neq \psi(p)\} \subset \{p \in M \mid H(p) \equiv 0\},$$

which proves the theorem. □

We cannot weaken the condition of H-semi-parallel in theorem 3.2.8 any further to being H-pseudo-parallel, because of the following proposition:

Proposition 3.2.11. *Let M be a Lagrangian submanifold of constant sectional curvature c. Then M is H-pseudo-parallel for the function $\phi = c$.*

Proof. Choose the function $\phi(p) = c$ for all $p \in M$. Since M has constant sectional curvature, we find that $(R^\perp - c\wedge^\perp) \equiv 0$ thus M is H-pseudo-parallel. □

Finally, when we look at Lagrangian surfaces of complex space forms, we may locally classify the H-semi-parallel Lagrangian surfaces.

Theorem 3.2.12. *Let M^2 be an H-semi-parallel surface. Then at every point $p \in M$, M is either minimal or flat [CL09].*

Proof. Since we are working with $n = 2$, $R^\perp = K\wedge^\perp$ with K the Gaussian curvature. Thus choosing a unit vector X orthogonal to JH, we obtain:

$$0 = \langle R^\perp(JH, X)H, JX\rangle = -K\langle (H \wedge JX)H, JX\rangle = K\|H\|^2,$$

thus at any point $p \in M$ either $K(p) = 0$ or $H(p) = 0$. □

However, the notion of H-pseudo-parallelity is useless for Lagrangian surfaces.

Proposition 3.2.13. *Let M^2 be a Lagrangian surface. Then M is H-pseudo-parallel for the function $\phi = K$ with K the Gaussian curvature.*

Proof. A surface always has curvature tensor $R = K\wedge$. So $(R - K\wedge) \equiv 0$ and thus M is H-pseudo-parallel for $\phi = K$. □

3.3 Umbilicity

In general, the simplest kind of manifolds besides the totally geodesic ones are the totally umbilical manifolds.

Definition 3.3.1. A submanifold is called *totally umbilical* if for any $\xi \in T_p^{\perp}M$, the shape operator A_ξ is a multiple of the identity. So there exists a constant λ_ξ such that for any $X \in T_pM$, we have

$$A_\xi X = \lambda_\xi X.$$

An equivalent definition in terms of the second fundamental form is that for any $\xi \in T_p^{\perp}M$, there is a constant λ_ξ such that for any $X, Y \in T_pM$,

$$\langle h(X,Y), \xi \rangle = \lambda_\xi \langle X, Y \rangle.$$

We can easily determine this multiple:

Proposition 3.3.2. *Let M^n be a totally umbilical submanifold. Then for any $\xi \in T_pM$, the shape operator is a multiple of the identity, and this multiple is $\lambda_\xi = \langle H, \xi \rangle$.*

Proof. From the definition of totally umbilical, we find that

$$\langle h(e_i, e_i), \xi \rangle = \lambda_\xi.$$

Taking the sum over all $1 \leq i \leq n$ and dividing by n gives us $\lambda_\xi = \langle H, \xi \rangle$. $\square$

As it turns out, in the Lagrangian setting, being totally umbilical is equivalent to being totally geodesic.

Theorem 3.3.3. *Let M^n be a Lagrangian submanifold. Then M is totally umbilical if and only if it is totally geodesic [CO74b].*

Proof. Let us first assume M is totally umbilical, i.e. for any $X, Y \in T_pM$,

$$A_{JX}Y = \langle H, JX \rangle Y.$$

Let X be any vector and choose a unit vector Y orthogonal to JH. We find that

$$\langle H, JX \rangle = \langle H, JX \rangle \langle Y, Y \rangle = \langle A_{JX}Y, Y \rangle = \langle A_{JY}X, Y \rangle = \langle H, JY \rangle \langle X, Y \rangle = 0,$$

and thus $A_{JX} \equiv 0$. This holds for any $X \in T_pM$, so $h(X,Y) = JA_{JX}Y = 0$ for any $X, Y \in T_pM$ and thus M is totally geodesic.

For the converse, note that if M is totally geodesic, it is minimal and thus

$$A_{JX}Y = -Jh(X,Y) = 0 = \langle H, JX \rangle Y,$$

for all $X, Y \in T_pM$, so M is totally umbilical. $\square$

Probably the simplest Lagrangian submanifolds besides the totally geodesic ones, are the *H-umbilical* submanifolds, which were introduced by Chen in [Che97a; Che97b].

Definition 3.3.4. A Lagrangian H-umbilical submanifold is a Lagrangian submanifold for which h takes the following form:

$$\begin{array}{lll} h(e_1,e_1)=\lambda Je_1 & h(e_2,e_2)=\ldots=h(e_n,e_n)=\mu Je_1 & \\ h(e_1,e_i)=\mu Je_i & h(e_i,e_j)=0 & 2\leq i\neq j\leq n, \end{array}$$

for some suitable functions λ and μ, with respect to some suitable orthonormal local frame field $\{e_1,\ldots,e_n\}$. An equivalent definition when $H\neq 0$ is given by

$$h(X,Y)=(\lambda-3\mu)\langle JX,H_1\rangle\langle JY,H_1\rangle H_1+\mu\left(\langle X,Y\rangle H_1+\langle JX,H_1\rangle JY+\langle JY,H_1\rangle JX\right),$$

where $H_1=H/\|H\|$ is the normalised mean curvature.

Property 3.3.5. If M is a non-minimal H-umbilical Lagrangian submanifold, then

(i) JH is an eigenvector of the shape operator A_H,

(ii) the restriction of A_H to $(JH)^\perp$ is a multiple of the identiy map,

(iii) if $\lambda=\mu$, then A_H is a multiple of the identity.

Proof. We start by pointing out that $H=\frac{\lambda+(n-1)\mu}{n}Je_1$, and thus $JH=\langle JH,e_1\rangle e_1$. For item (i), we calculate A_HJH:

$$A_HJH=Jh(JH,JH)=\langle JH,e_1\rangle^2Jh(e_1,e_1)=-\lambda\langle JH,e_1\rangle^2e_1=-\lambda\langle JH,e_1\rangle JH.$$

For item (ii), let $X\in(JH)^\perp$, then $X\perp e_1$. We calculate A_HX:

$$\begin{aligned} A_HX=Jh(JH,X)&=\langle JH,e_1\rangle J\sum_{i=2}^{n}\langle X,e_i\rangle h(e_1,e_i)=-\mu\langle JH,e_1\rangle\sum_{i=2}^{n}\langle X,e_i\rangle e_i\\ &=-\mu\langle JH,e_1\rangle X. \end{aligned}$$

Now, if $\lambda=\mu$, then $H=\lambda Je_1$. By item (i) and (ii) we find that for any $X\in T_pM$,

$$A_HX=-\lambda\langle JH,e_1\rangle X=\|H\|^2X,$$

so A_H is a multiple of the identity. □

Another notion of umbilicity weaker than totally umbilical exists:

Definition 3.3.6. A submanifold is called *pseudo-umbilical* if the shape operator A_H is a multiple of the identity. So there exists a constant λ such that for any $X\in T_pM$, we have

$$A_HX=\lambda X.$$

An equivalent definition in terms of the second fundamental form is that there is a constant λ such that for any $X,Y\in T_pM$,

$$\langle h(X,Y),H\rangle=\lambda\langle X,Y\rangle.$$

Again, we can easily determine this multiple:

Proposition 3.3.7. *Let M^n be a pseudo-umbilical submanifold. Then the shape operator A_H is a multiple of the identity, and this multiple is $\|H\|^2$.*

Proof. From the definition of pseudo-umbilical, we find that

$$\langle h(e_i, e_i), H\rangle = \lambda.$$

Taking the sum over all $1 \leq i \leq n$ and dividing by n gives us $\lambda = \|H\|^2$. □

Proposition 3.3.8. *Let M be a minimal submanifold. Then M is pseudo-umbilical.*

Proof. If M is minimal, then $A_H X = 0 = \|H\|^2 X$ for all $X \in T_p M$. □

The following gives a relation between H-umbilicity and pseudo-umbilicity:

Proposition 3.3.9. *Let M^n be a H-umbilical Lagrangian submanifold. Then M is pseudo-umbilical if and only if at any point p, either $H(p) = 0$ or $\lambda(p) = \mu(p)$.*

Proof. First assume M is pseudo-umbilical. If M is minimal at p there is nothing to prove. So assume $H(p) \neq 0$, then by properties (i) and (ii) of property 3.3.5, we find

$$\lambda\langle JH, e_1\rangle JH = \mu\langle JH, e_1\rangle JH,$$

so $\lambda(p) = \mu(p)$.

Conversely, assume that at any point $p \in M$, either $H(p) = 0$ or $\lambda(p) = \mu(p)$. If $H(p) = 0$ then M is trivially pseudo-umbilical at p. If $\lambda(p) = \mu(p)$, then by item (iii) of property 3.3.5 M is pseudo-umbilical at p. So M is a pseudo-umbilical submanifold. □

In [Hua97] it is claimed that H-umbilical submanifolds of $\mathbb{C}P^n$ are pseudo-umbilical. However, making use of the previous lemma, we can provide a counterexample.

Example 3.3.10. Let $\tilde{M}^n(4\tilde{c})$ be the complex projective space $\mathbb{C}P^n(4\tilde{c})$ and let $M^n(c)$ be a simply-connected open portion of the Riemannian n-sphere $S^n(c)$ $(c > \tilde{c})$. Then M admits a Lagrangian H-umbilical isometric immersion into $\tilde{M}$ such that

$$\begin{aligned} h(e_1, e_1) &= 2\sqrt{c - \tilde{c}} J e_1, & h(e_i, e_i) &= \sqrt{c - \tilde{c}} J e_1 \\ h(e_1, e_i) &= \sqrt{c - \tilde{c}} J e_i, & h(e_i, e_j) &= 0, \end{aligned}$$

for some orthonormal local frame field $\{e_1, \ldots, e_n\}$ on M, where $2 \leq j \neq k \leq n$. Then M is not minimal and $\lambda \neq \mu$. In particular, by proposition 3.3.9, M is not pseudo-umbilical [Che97b].

Similarly, we provide counterexamples for $\mathbb{C}^n$ and $\mathbb{C}H^n$:

Example 3.3.11. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$. The map

$$w : S^n \to \mathbb{C}^n : (y_0, \ldots, y_n) \mapsto \frac{1 + iy_0}{1 + y_0^2}(y_1, \ldots, y_n),$$

is a Lagrangian H-umbilical immersion called the *Whitney n-sphere*, which satisfies $\lambda = 3\mu \neq 0$. In fact, up to homothetic transformations, it is the only Lagrangian H-umbilical submanifold in $\mathbb{C}^n$ satisfying $\lambda = 3\mu$. The second fundamental form of the Whitney sphere is of the form

$$h(X,Y) = \frac{n}{n+2}\left(\langle X,Y\rangle H + \langle JX,H\rangle JY + \langle JY,H\rangle JX\right),$$

and in fact, the only Lagrangian submanifolds of $\mathbb{C}^n$ to have this form are the totally geodesic ones and the Whitney Sphere.

Since the Whitney sphere has nonzero mean curvature it is not minimal, so by proposition 3.3.9 it is not pseudo-umbilical [BCM95; Che97a; CU93; DS04; RU98; Wei77].

Example 3.3.12. Let $\tilde{M}^n(4\tilde{c})$ be the complex hyperbolic space $\mathbb{C}H^n(4\tilde{c})$ and let $M^n(c)$ be a real space form ($c > \tilde{c}$). Then M admits (at least locally) a Lagrangian H-umbilical isometric immersion into $\tilde{M}$ such that

$$\begin{aligned} h(e_1,e_1) &= 2\sqrt{c-\tilde{c}}Je_1, & h(e_i,e_i) &= \sqrt{c-\tilde{c}}Je_1 \\ h(e_1,e_i) &= \sqrt{c-\tilde{c}}Je_i, & h(e_i,e_j) &= 0, \end{aligned}$$

for some orthonormal local frame field $\{e_1,\ldots,e_n\}$ on M, where $2 \le j \ne k \le n$. Then M is not minimal and $\lambda \ne \mu$. Thus by proposition 3.3.9, M cannot be pseudo-umbilical [Che97b].

Definition 3.3.13. A *Lagrangian-umbilical* submanifold is a Lagrangian H-umbilical submanifold for which $\lambda = \mu$.

Proposition 3.3.14. *A Lagrangian-umbilical submanifold is both H-umbilical and pseudo-umbilical.*

Proof. This follows from the definition and proposition 3.3.9. □

The following is an example of a non-minimal Lagrangian-umbilical submanifold:

Example 3.3.15. For $a \in \mathbb{R}_0$, let

$$F(s) = \int^s e^{-ia\log(t)}dt,$$

where $\int^s f(t)dt$ denotes an antiderivative of $f(s)$. Let $\iota : S^{n-1}(1) \to \mathbb{E}^n$ be the unit hypersphere centred at the origin. Then the complex extensor (see the article referenced below for a definition) $F \otimes \iota : I \times S^{n-1}(1) \to \mathbb{C}^n$ is a H-umbilical Lagrangian submanifold M with $\lambda = \mu = -a/s$. So M is not minimal, but by the previous lemma it is pseudo-umbilical [Che97a].

Next, we show there is a link between pseudo-umbilical Lagrangian submanifolds and H-pseudo-parallel Lagrangian submanifolds. We first introduce the following lemma:

Lemma 3.3.16. *Let M be a pseudo-umbilical Lagrangian submanifold. Then for all X, Y,*

$$[A_{JX}, A_{JY}]JH = 0.$$

Proof. This follows from the symmetry of the shape operator:

$$-A_{JX}A_{JY}JH = A_{JX}A_HY = \|H\|^2 A_{JX}Y = \|H\|^2 A_{JY}X = A_{JY}A_HX = -A_{JY}A_{JX}JH,$$

which proves the lemma. □

Theorem 3.3.17. *If M is a pseudo-umbilical Lagrangian submanifold, then M is H-pseudo-parallel for $\phi = \tilde{c}$. Moreover, if the ambient manifold is $\mathbb{C}^n$, then it is H-semi-parallel.*

Proof. Take $\phi = \tilde{c}$. Since M is pseudo-umbilical, we use (2.1.7) and the previous lemma to find

$$(R(X,Y) - \tilde{c}(X \wedge Y))JH = [A_{JX}, A_{JY}]JH = 0,$$

so M is H-pseudo-parallel. If the ambient manifold is $\mathbb{C}^n$, then $\phi = \tilde{c} = 0$ so M is H-semi-parallel. □

Theorem 3.3.18. *If M is a Lagrangian submanifold of $\mathbb{C}P^n$ or $\mathbb{C}H^n$ (i.e. $\tilde{c} \neq 0$), then M is minimal if and only if M is pseudo-umbilical and H-semi-parallel.*

Proof. If M is minimal, then M is trivially pseudo-umbilical and H-semi-parallel. Conversely, assume M is both pseudo-umbilical and H-semi-parallel. Let X be a unit vector orthogonal to JH. By equation (2.1.7) we find that

$$0 = \langle R(X,JH)JH, X\rangle = \tilde{c}\langle (X \wedge JH)JH, X\rangle = \tilde{c}\langle JH, JH\rangle\langle X, X\rangle = \tilde{c}\|H\|^2,$$

thus M is minimal. □

In the case of Lagrangian surfaces, we find that pseudo-umbilicity is a stronger condition than H-umbilicity:

Theorem 3.3.19. *Let M^2 be a pseudo-umbilical Lagrangian surface. Then M is H-umbilical.*

Proof. We will have to prove this differently depending on whether M is minimal at a point p or not. First, let $p \in M$ and suppose that $H(p) = 0$. Let $\{e_1, \ldots, e_n\}$ be a canonical basis, then $C_{111} = -C_{122} = \lambda_1$ and $C_{112} = -C_{222} = 0$. Thus we have that

$$\begin{aligned} h(e_1,e_1) &= \lambda_1 Je_1, \\ h(e_1,e_2) &= -\lambda_1 Je_2, \\ h(e_2,e_2) &= -\lambda_1 Je_1, \end{aligned}$$

so M is H-umbilical at p with $\lambda = -\mu$.

Now assume $H(p) \neq 0$. Then choose $e_1 = -JH/\|H\|$, so $A_H = \|H\| A_{Je_1}$. Then

$$\begin{aligned} h(e_1,e_1) &= JA_{Je_1}e_1 = \frac{1}{\|H\|}JA_He_1 = \|H\|Je_1, \\ h(e_1,e_2) &= JA_{Je_1}e_2 = \frac{1}{\|H\|}JA_He_2 = \|H\|Je_2, \\ h(e_2,e_2) &= 2H - h(e_1,e_1) = \|H\|Je_1, \end{aligned}$$

so M is H-umbilical at p with $\lambda = \mu = \|H\|$. □

3.4 Second fundamental form

Rather than take the trace of the second fundamental form h and then differentiate, we can simply work with h itself. We give the following definitions:

Definition 3.4.1. We define the following conditions related to the second fundamental form h:

Name	**Condition**
totally geodesic	$h \equiv 0$
parallel	$\bar{\nabla} h \equiv 0$
semi-parallel	$\bar{R} \cdot h \equiv 0$
pseudo-parallel	$(\bar{R} - \phi\wedge) \cdot h \equiv 0$

Proposition 3.4.2. *Every condition in the above table implies the next:*

$$h \equiv 0 \implies \bar{\nabla} h \equiv 0 \implies \bar{R} \cdot h \equiv 0 \implies (\bar{R} - \phi\wedge) h \equiv 0.$$

Remark 3.4.3. Parallel submanifolds were introduced by Vilms in [Vil72] and elaborated by Ferus in [Fer74; Fer80]. Semi-parallel submanifolds were defined by Deprez in [Dep85; Dep86] and pseudo-parallel submanifolds by Asperti, Lobos and Mercuri in [ALM02].

Proposition 3.4.4. *Let M be a parallel Lagrangian submanifold. Then M is H-parallel.*

Proof. We have to prove that for any $X \in T_pM$, $\nabla^\perp_X H = 0$. So take $X \in T_pM$ and consider

$$0 = \frac{1}{n}\sum_{i=1}^{n} (\bar{\nabla}_X h)(e_i, e_i) = \frac{1}{n}\sum_{i=1}^{n} \left(\nabla^\perp_X h(e_i, e_i) - 2h(\nabla_X e_i, e_i)\right) = \nabla^\perp_X H - \frac{2}{n}\sum_{i=1}^{n} h(\nabla_X e_i, e_i).$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM. It suffices to prove that the summation in the second term vanishes. Note that from the properties of the Levi-Civita connection, we find

$$\begin{aligned} 0 &= X\langle e_i, e_j\rangle = \langle \nabla_X e_i, e_j\rangle + \langle \nabla_X e_j, e_i\rangle, \\ 0 &= X\langle e_i, e_i\rangle = 2\langle \nabla_X e_i, e_i\rangle. \end{aligned}$$

Applying these properties, the summation becomes

$$\begin{aligned} \sum_{i=1}^{n} h(\nabla_X e_i, e_i) &= \sum_{i,j} \langle \nabla_X e_i, e_j\rangle h(e_i, e_j) \\ &= \sum_{i<j} \langle \nabla_X e_i, e_j\rangle h(e_i, e_j) + \sum_{i>j} \langle \nabla_X e_i, e_j\rangle h(e_i, e_j) \\ &= \sum_{i<j} \langle \nabla_X e_i, e_j\rangle h(e_i, e_j) + \sum_{i<j} \langle \nabla_X e_j, e_i\rangle h(e_j, e_i) \\ &= \sum_{i<j} \langle \nabla_X e_i, e_j\rangle h(e_i, e_j) - \sum_{i<j} \langle \nabla_X e_i, e_j\rangle h(e_i, e_j) = 0. \end{aligned}$$

So we have that $\nabla^\perp_X H = 0$ for all $X \in T_pM$, so M is H-parallel. $\square$

The function $\phi \in \mathcal{F}(M)$ in the definition of pseudo-parallelity is unique in some sense.

Theorem 3.4.5. *Let M be a pseudo-parallel Lagrangian submanifold, for both the functions ϕ and ψ. Then $\phi = \psi$ on $M \backslash \{p \in M \mid h_p \equiv 0\}$ [CL09].*

Proof. We rewrite the condition of pseudo-parallelity

$$\begin{aligned}
0 =& \bar{R}(X,Y) \cdot h(U,V) - \phi(X \wedge Y \cdot h(U,V)) \\
=& R^{\perp}(X,Y)h(U,V) - h(R(X,Y)U,V) - h(U,R(X,Y)V) \\
& - \phi(\langle Y,U\rangle h(X,V) - \langle X,U\rangle h(Y,V) + \langle Y,V\rangle h(U,X) - \langle X,V\rangle h(U,Y)).
\end{aligned}$$

If M is pseudo-parallel for both ϕ and ψ, then we find

$$(\phi - \psi)(\langle Y,U\rangle h(X,V) - \langle X,U\rangle h(Y,V) + \langle Y,V\rangle h(U,X) - \langle X,V\rangle h(U,Y)) = 0.$$

Let $X = U = V$ be a unit vector and Y a unit vector orthogonal to X, then we get

$$(\phi - \psi)h(X,Y) = 0,$$

and taking $X = U$, $Y = V$ both unit vectors orthogonal to each other, we get

$$(\phi - \psi)(h(X,X) - h(Y,Y)) = 0.$$

Now, let $p \in M$ such that $\phi(p) \neq \psi(p)$. Then $h(X,Y) = 0$ and $h(X,X) = h(Y,Y)$, so

$$\langle h(X,X), JY\rangle = \langle h(X,Y), JX\rangle = 0,$$

which means that $h(X,X)$ lies completely in the direction of JX. Thus we get

$$h(X,X) = \langle h(X,X), JX\rangle JX = \langle h(Y,Y), JX\rangle JX = \langle h(X,Y), JY\rangle JX = 0,$$

so $h_p \equiv 0$. Consequently,

$$\{p \in M \mid \phi(p) \neq \psi(p)\} \subset \{p \in M \mid h_p \equiv 0\},$$

which proves the theorem. □

A link can be found between pseudo-parallel Lagrangian submanifolds and H-semi-parallel Lagrangian submanifolds.

Proposition 3.4.6. *Let M be a pseudo-parallel Lagrangian submanifold. Then M is H-semi-parallel [CL09].*

Proof. Let $Z \in T_pM$ and choose an orthonormal basis $\{e_1, \ldots, e_n\}$ that diagonalises A_{JZ}, so in particular $h(Z, e_i) = \lambda_i J e_i$ for certain λ_i. Then for any $X, Y \in T_pM$ we obtain

$$\begin{aligned}
\langle R^{\perp}(X,Y)H, JZ\rangle &= \frac{1}{n}\sum_{i=1}^{n}\langle R^{\perp}(X,Y)h(e_i,e_i), JZ\rangle \\
&= \frac{2}{n}\sum_{i=1}^{n}(\phi\langle h((X \wedge Y)e_i, e_i), JZ\rangle - \langle h(R(X,Y)e_i, e_i), JZ\rangle) \\
&= \frac{2}{n}\sum_{i=1}^{n}(\phi\langle h(Z,e_i), J(X \wedge Y)e_i\rangle - \langle h(Z,e_i), JR(X,Y)e_i\rangle) \\
&= \frac{2}{n}\sum_{i=1}^{n}\lambda_i(\phi\langle e_i, (X \wedge Y)e_i\rangle - \langle e_i, R(X,Y)e_i\rangle) = 0,
\end{aligned}$$

because of the symmetries of curvature tensors. So M is H-semi-parallel. □

However, the notion of pseudo-parallelity is not very useful in the Lagrangian case, as we shall prove in the next two theorems.

Theorem 3.4.7. *A pseudo-parallel Lagrangian submanifold M^n of a complex space form $\tilde{M}^n(4\tilde{c})$ of dimension $n \geq 3$ is semi-parallel [DVV09].*

Proof. Let $p \in M^n$ be a point such that $\phi(p) \neq 0$. We will first play around a bit with the symmetry of the condition of pseudo-parallelity: we have that

$$\langle \bar{R}(X,Y) \cdot h(U,V) - \phi(X \wedge Y \cdot h)(U,V), JW \rangle = 0,$$

for all X, Y, U, V, W, and in particular this is symmetric in V and W. Now, we prove that $\langle \bar{R}(X,Y) \cdot h(U,V), JW \rangle$ on its own is symmetric in V and W. This can be seen by writing $\bar{R} \cdot h$ in full:

$$\langle R^{\perp}(X,Y)h(U,V), JW \rangle - \langle h(R(X,Y)U,V), JW \rangle - \langle h(U, R(X,Y)V), JW \rangle.$$

Since the cubic form is totally symmetric, the middle term is symmetric in V and W. For the first and last term, consider:

$$\begin{aligned} \langle R^{\perp}(X,Y)h(U,V), JW \rangle &= -\langle JR^{\perp}(X,Y)h(U,V), W \rangle = -\langle R(X,Y)Jh(U,V), W \rangle \\ &= \langle R(X,Y)W, Jh(U,V) \rangle = -\langle JR(X,Y)W, h(U,V) \rangle \\ &= -\langle h(U, R(X,Y)W), JV \rangle, \end{aligned}$$

so the first and last terms together are also symmetric in V and W. Since M is pseudo-parallel, we have that $\langle (X \wedge Y) \cdot h(U,V), JW \rangle$ is then symmetric in V and W. In other words,

$$\begin{aligned} &\langle Y,U \rangle \langle h(X,V), JW \rangle - \langle X,U \rangle \langle h(Y,V), JW \rangle \\ &+ \langle Y,V \rangle \langle h(X,U), JW \rangle - \langle X,V \rangle \langle h(Y,U), JW \rangle, \end{aligned}$$

is symmetric in V and W. But clearly the first two terms are both symmetric in V and W too, so the last two terms together must also be symmetric in V and W:

$$\begin{aligned} &\langle Y,V \rangle \langle h(X,U), JW \rangle - \langle X,V \rangle \langle h(Y,U), JW \rangle \\ &= \langle Y,W \rangle \langle h(X,U), JV \rangle - \langle X,W \rangle \langle h(Y,U), JV \rangle. \end{aligned}$$

Take $X = U = V$ and Y, W orthogonal to X, we get:

$$-\langle X,X \rangle \langle h(X,Y), JW \rangle = \langle Y,W \rangle \langle h(X,X), JX \rangle. \tag{3.4.1}$$

Let $\{e_1, \ldots, e_n\}$ be a canonical basis. Then taking $X = e_1, Y = e_i, W = e_j$ with $i, j \neq 1$ in (3.4.1) gives

$$\langle h(e_1, e_i), Je_j \rangle = -\lambda_1 \langle e_i, e_j \rangle = -\lambda_1 \delta_{ij},$$

and by taking $X = e_i$, with $i \neq 1$, and $Y = W = e_1$ we get

$$\langle h(e_i, e_i), Je_i \rangle = -\langle e_i, e_i \rangle \langle h(e_1, e_1), Je_i \rangle = 0.$$

By the linearity and total symmetry of the cubic form, we get that $\langle h(e_i, e_j), Je_k \rangle = 0$ for all e_i, e_j, e_k with $i, j, k \neq 1$. So h takes the following form:

$$\begin{aligned} h(e_1, e_1) &= \lambda_1 Je_1, & h(e_1, e_i) &= -\lambda_1 Je_i \\ h(e_i, e_j) &= -\lambda_1 \delta_{ij} Je_1, & 2 &\leq i, j \leq n. \end{aligned}$$

In fact, this would mean M^n is a H-umbilical Lagrangian submanifold with $\mu = -\lambda$. If $\lambda_1 = 0$, then h vanishes at p. So now assume $\lambda_1 \neq 0$. Since $n \geq 3$, the vectors e_2 and e_3 exist. Now, note that

$$\begin{aligned} R(e_1, e_2)e_1 &= \tilde{c}(e_1 \wedge e_2)e_1 + [A_{Je_1}, A_{Je_2}]e_1 \\ &= -\tilde{c}e_2 + \lambda_1^2 e_2 + \lambda_1^2 e_2 \\ &= -(\tilde{c} - 2\lambda_1^2)e_2, \end{aligned}$$

and similarly

$$\begin{aligned} R(e_1, e_2)e_2 &= \tilde{c}(e_1 \wedge e_2)e_2 + [A_{Je_1}, A_{Je_2}]e_2 \\ &= \tilde{c}e_1 - \lambda_1^2 e_1 - \lambda_1^2 e_1 \\ &= (\tilde{c} - 2\lambda_1^2)e_1. \end{aligned}$$

Hence

$$\begin{aligned} 0 &= (\bar{R}(e_1, e_2) \cdot h)(e_2, e_2) + \phi(p)(e_1 \wedge e_2 \cdot h)(e_2, e_2) \\ &= -JR(e_1, e_2)Jh(e_2, e_2) - 2h(R(e_1, e_2)e_2, e_2) + 2\phi(p)h(e_1, e_2) \\ &= -\lambda_1 JR(e_1, e_2)e_1 - 2(\tilde{c} - 2\lambda_1^2)h(e_1, e_2) - 2\lambda_1\phi(p)Je_2 \\ &= \lambda_1(3(\tilde{c} - 2\lambda_1^2) - 2\phi(p))Je_2, \end{aligned}$$

thus for ϕ we find the following value at p:

$$\phi(p) = \frac{3}{2}(\tilde{c} - 2\lambda_1^2), \tag{3.4.2}$$

Similarly, from

$$\begin{aligned} 0 &= (\bar{R}(e_1, e_2) \cdot h)(e_2, e_3) + \phi(p)(e_1 \wedge e_2 \cdot h)(e_2, e_3) \\ &= -h(R(e_1, e_2)e_2, e_3) + \phi(p)h(e_1, e_3) \\ &= \lambda_1((\tilde{c} - 2\lambda_1^2) - \phi(p))Je_3, \end{aligned}$$

we find

$$\phi(p) = (\tilde{c} - 2\lambda_1^2). \tag{3.4.3}$$

Combining equations (3.4.2) and (3.4.3) we obtain $\phi(p) = 0$. We conclude that either $\phi(p) = 0$ or $h_p \equiv 0$. Either way, $\bar{R} \cdot h$ vanishes at p so M is semi-parallel. □

Theorem 3.4.8. *Let M^2 be a H-semi-parallel Lagrangian surface. Then M is pseudo-parallel with $\phi = \frac{3}{2}K$, where K is the Gaussian curvature of M [CL09].*

Proof. Take X a unit tangent vector orthogonal to JH. Since M is H-semi-parallel we obtain

$$\begin{aligned} 0 &= \langle R^{\perp}(JH,X)H, JX\rangle = -\langle R(JH,X)JH, X\rangle = -\langle K(JH\wedge X)JH, X\rangle \\ &= K\langle JH, (JH\wedge X)X\rangle = K\|H\|^2, \end{aligned}$$

thus at every point $p \in M$, $K(p) = 0$ or $H(p) = 0$. If $K(p) = 0$, then $R_p \equiv 0$ and M is semi-parallel at p, so it is pseudo-parallel with $\phi(p) = 0$. Now assume $K(p) \neq 0$, and thus $H(p) = 0$. Then h in terms of the canonical basis becomes:

$$\begin{aligned} h(e_1, e_1) &= \lambda J e_1, \\ h(e_1, e_2) &= -\lambda J e_2, \\ h(e_2, e_2) &= -\lambda J e_1. \end{aligned}$$

Consider the condition of being pseudo-parallel with $\phi = \frac{3}{2}K$:

$$\begin{aligned} & R^{\perp}(X,Y)h(U,V) - h(R(X,Y)U,V) - h(U,R(X,Y)V) \\ & \quad + \frac{3}{2}K(\langle Y,U\rangle h(X,V) - \langle X,U\rangle h(Y,V) + \langle Y,V\rangle h(U,X) - \langle X,V\rangle h(U,Y)) \\ & = JK(X\wedge Y)Jh(U,V) - Kh((X\wedge Y)U,V) - Kh(U,(X\wedge Y)V) \\ & \quad + \frac{3}{2}K(\langle Y,U\rangle h(X,V) - \langle X,U\rangle h(Y,V) + \langle Y,V\rangle h(U,X) - \langle X,V\rangle h(U,Y)) \\ & = K\langle Y, Jh(U,V)\rangle JX - K\langle X, Jh(U,V)\rangle JY \\ & \quad + \frac{1}{2}K(\langle Y,U\rangle h(X,V) - \langle X,U\rangle h(Y,V) + \langle Y,V\rangle h(U,X) - \langle X,V\rangle h(U,Y)). \end{aligned}$$

Since $K(p) \neq 0$, we end up with

$$\begin{aligned} & \langle Y, Jh(U,V)\rangle JX - \langle X, Jh(U,V)\rangle JY \\ & \quad + \frac{1}{2}(\langle Y,U\rangle h(X,V) - \langle X,U\rangle h(Y,V) + \langle Y,V\rangle h(U,X) - \langle X,V\rangle h(U,Y)). \end{aligned}$$

We have to show that this vanishes for any X, Y, U, V. Because of linearity, antisymmetry in X and Y and symmetry in U and V, it suffices to show that this is true for 3 cases: $(X,Y,U,V) = (e_1,e_2,e_1,e_1)$, $(X,Y,U,V) = (e_1,e_2,e_1,e_2)$ and $(X,Y,U,V) = (e_1,e_2,e_2,e_2)$.

In the case $(X,Y,U,V) = (e_1,e_2,e_1,e_1)$ we find

$$\begin{aligned} & \langle e_2, Jh(e_1,e_1)\rangle Je_1 - \langle e_1, Jh(e_1,e_1)\rangle Je_2 \\ & \quad + \frac{1}{2}(\langle e_2,e_1\rangle h(e_1,e_1) - \langle e_1,e_1\rangle h(e_2,e_1) + \langle e_2,e_1\rangle h(e_1,e_1) - \langle e_1,e_1\rangle h(e_1,e_2)) \\ & = -\lambda Je_2 + \frac{1}{2}(\lambda Je_2 + \lambda Je_2) = 0. \end{aligned}$$

In the case $(X,Y,U,V) = (e_1,e_2,e_1,e_2)$ we find

$$\begin{aligned} & \langle e_2, Jh(e_1,e_2)\rangle Je_1 - \langle e_1, Jh(e_1,e_2)\rangle Je_2 \\ & \quad + \frac{1}{2}(\langle e_2,e_1\rangle h(e_1,e_2) - \langle e_1,e_1\rangle h(e_2,e_2) + \langle e_2,e_2\rangle h(e_1,e_1) - \langle e_1,e_2\rangle h(e_1,e_2)) \\ & = -\lambda Je_1 + \frac{1}{2}(\lambda Je_1 + \lambda Je_1) = 0. \end{aligned}$$

Finally, in the case $(X,Y,U,V) = (e_1, e_2, e_2, e_2)$ we find

$$\begin{aligned}
&\langle e_2, Jh(e_2,e_2)\rangle Je_1 - \langle e_1, Jh(e_2,e_2)\rangle Je_2 \\
&+ \frac{1}{2}(\langle e_2,e_2\rangle h(e_1,e_2) - \langle e_1,e_2\rangle h(e_2,e_2) + \langle e_2,e_2\rangle h(e_2,e_1) - \langle e_1,e_2\rangle h(e_2,e_2)) \\
&= \lambda Je_2 + \frac{1}{2}(-\lambda Je_2 - \lambda Je_2) = 0.
\end{aligned}$$

Thus M is pseudo-parallel with $\phi = \frac{3}{2}K$. □

Corollary 3.4.9. *A Lagrangian surface is pseudo-parallel if and only if it is H-semi-parallel.*

So if $n = 2$, pseudo-parallelity is equivalent to H-semi-parallelity, and if $n \geq 3$ pseudo-parallelity is equivalent to semi-parallelity. Thus the notion of pseudo-parallelity is always equivalent to a previously established notion.

3.5 Cubic form

The cubic form C and the second fundamental form h are closely related. Just like for the second fundamental form, we can study the derivatives of C.

Definition 3.5.1. We define the following conditions related to the cubic form C:

Name	**Condition**
totally geodesic	$C \equiv 0$
parallel	$\nabla C \equiv 0$
semi-parallel	$R \cdot C \equiv 0$
pseudo-parallel cubic form	$(R - \phi\wedge) \cdot C \equiv 0$

Remark 3.5.2. The first 3 conditions are identical to the conditions $h \equiv 0$, $\bar{\nabla} h \equiv 0$ and $\bar{R} \cdot h \equiv 0$ respectively. The last condition is not equivalent to $(\bar{R} - \phi\wedge) \cdot h \equiv 0$, but rather to $(\bar{R} - \phi\bar{\wedge}) \cdot h \equiv 0$ instead, the condition of Lagrangian pseudo-parallelity.

Proposition 3.5.3. *Every condition in the above table implies the next:*

$$C \equiv 0 \implies \nabla C \equiv 0 \implies R \cdot C \equiv 0 \implies (R - \phi\wedge)C \equiv 0.$$

We also have that the derivatives of h and C are very closely related.

Proposition 3.5.4. *Let $k \in \mathbb{N}$. Then $(\nabla^k_{X_1,\ldots,X_k} C)(Y,Z,W) = \langle (\bar{\nabla}^k_{X_1,\ldots,X_n} h)(Y,Z), JW\rangle$.*

Corollary 3.5.5. *Let $k \in \mathbb{N}$. Then $\bar{\nabla}^k h = 0$ if and only if $\nabla^k C = 0$.*

Again, we want to prove that the function ϕ in the definition of pseudo-parallel cubic form is unique in some sense.

Theorem 3.5.6. *Let M have pseudo-parallel cubic form for both the functions ϕ and ψ. Then $\phi = \psi$ on $M \backslash \{p \in M \mid C_p \equiv 0\}$.*

Proof. If M is both pseudo-parallel for ϕ and ψ, then we find

$$(\phi-\psi)(C((X\wedge Y)U,V,W)+C(U,(X\wedge Y)V,W)+C(U,V,(X\wedge Y)W))=0.$$

Let X be any unit vector and take a unit vector $Y=U=V=W$ orthogonal to X, then we get

$$0=(\phi-\psi)C((X\wedge Y)Y,Y,Y)=(\phi-\psi)C(X,Y,Y).$$

Taking $X=U=V$, and $Y=W$ unit and orthogonal to X, we get

$$\begin{aligned}0&=(\phi-\psi)(C((X\wedge Y)X,X,Y)+2C(X,X,(X\wedge Y)Y)\\&=(\phi-\psi)(C(X,X,X)-2C(X,X,Y)\\&=(\phi-\psi)C(X,X,X).\end{aligned}$$

Now, let $p\in M$ such that $\phi(p)\neq\psi(p)$. Then by linearity, $C_p\equiv 0$. Consequently,

$$\{p\in M\mid \phi(p)\neq\psi(p)\}\subset\{p\in M\mid C_p\equiv 0\},$$

which proves the theorem. □

Proposition 3.5.7. *If M has constant sectional curvature c, then it has pseudo-parallel cubic form for the function $\phi=c$. Moreover, if M is flat, then it is semi-parallel.*

Proof. The curvature tensor of M is of the form $R(X,Y)=c(X\wedge Y)$ (with $c=0$ if M is flat). Taking $\phi=c$ then gives $(R-c\wedge)\equiv 0$ so M has pseudo-parallel cubic form. In particular, if M is semi-parallel, it has pseudo-parallel cubic form for $\phi=c=0$ so it is semi-parallel. □

Now we know that semi-parallelity implies both H-semi-parallelity and pseudo-parallel cubic form. We investigate what happens if a Lagrangian submanifolds has both of these weaker conditions.

Theorem 3.5.8. *If M is H-semi-parallel and has pseudo-parallel cubic form at p, then it is semi-parallel or minimal at p.*

Proof. Choose a unit vector X orthogonal to JH, and let $\{e_1,\dots,e_n\}$ be an orthonormal basis that diagonalises A_{JX}, so in particular $h(X,e_i)=\lambda_i e_i$ for some λ_i. Then

$$\begin{aligned}0&=\langle JX,R^\perp(X,JH)H\rangle=-\langle H,JR(X,JH)X\rangle\\&=\frac{1}{n}\sum_{i=1}^{n}\langle h(e_i,e_i),JR(X,JH)X\rangle=\frac{1}{n}\sum_{i=1}^{n}C(e_i,e_i,R(X,JH)X)\\&=\frac{1}{n}\sum_{i=1}^{n}(-2C(R(X,JH)e_i,e_i,X)+\phi C(e_i,e_i,(X\wedge JH)X)+2\phi C((X\wedge JH)e_i,e_i,X))\\&=\frac{1}{n}\sum_{i=1}^{n}(-2\langle h(X,e_i),JR(X,JH)e_i\rangle-\phi C(e_i,e_i,JH)+2\phi\langle h(X,e_i),J(X\wedge JH)e_i\rangle)\\&=\frac{1}{n}\sum_{i=1}^{n}(-2\lambda_i\langle R(X,JH)e_i,e_i\rangle+\phi\langle h(e_i,e_i),H\rangle+2\lambda_i\phi\langle (X\wedge JH)e_i,e_i\rangle)\\&=\phi\|H\|^2.\end{aligned}$$

So either $\phi(p)=0$ or $H(p)=0$. Thus M is either minimal or semi-parallel at p. □

Theorem 3.5.9. *Let M be a H-umbilical Lagrangian submanifold. Then M has pseudo-parallel cubic form for the function $\phi = \tilde{c} + \mu(\lambda - \mu)$.*

Proof. If we take $\phi = \tilde{c} - \mu(\lambda - \mu)$, then

$$R(X,Y) - (\tilde{c} + \mu(\lambda - \mu))(X \wedge Y) = \mu(\mu - \lambda)(X \wedge Y) + [A_{JX}, A_{JY}].$$

Because the condition of having pseudo-parallel cubic form is equivalent to that of Lagrangian pseudo-parallel and because J and R commute for derivatives, it suffices to apply this operator to $Jh(U,V)$ and show that the result always vanishes. By linearity we only have to prove that it vanishes for vectors belonging to a canonical basis $\{e_1, \ldots, e_n\}$, so let us take $X = e_i, Y = e_j, U = e_a, V = e_b$ with $i \leq j$ and $a \leq b$. Clearly everything vanishes if $i = j$, so exploiting the symmetry we have 8 cases to prove:

(i) $1 = i \neq j$, $1 = a = b$,

(ii) $1 = i \neq j$, $1 = a \neq b$,

(iii) $1 = i \neq j$, $1 \neq a = b$,

(iv) $1 = i \neq j$, $1 \neq a \neq b$,

(v) $1 \neq i \neq j$, $1 = a = b$,

(vi) $1 \neq i \neq j$, $1 = a \neq b$,

(vii) $1 \neq i \neq j$, $1 \neq a = b$,

(viii) $1 \neq i \neq j$, $1 \neq a \neq b$.

Let us first write out the condition in terms of this basis:

$$\begin{aligned}
&(\mu(\mu - \lambda)(e_i \wedge e_j) + [A_{Je_i}, A_{Je_j}]) \cdot Jh(e_a, e_b) \\
&= (\mu(\mu - \lambda)\left((e_i \wedge e_j)Jh(e_a, e_b) - Jh((e_i \wedge e_j)e_a, e_b) - Jh(e_a, (e_i \wedge e_j)e_b)\right) \\
&\quad + [A_{Je_i}, A_{Je_j}]Jh(e_a, e_b) - Jh([A_{Je_i}, A_{Je_j}]e_a, e_b) - Jh(e_a, [A_{Je_i}, A_{Je_j}]e_b) \\
&= \mu(\mu - \lambda)\left(-(e_i \wedge e_j)A_{Je_a}e_b + \delta_{aj}A_{Je_i}e_b - \delta_{ai}A_{Je_j}e_b + \delta_{bj}A_{Je_i}e_a - \delta_{bi}A_{Je_j}e_a\right) \\
&\quad - A_{Je_i}A_{Je_j}A_{Je_a}e_b + A_{Je_j}A_{Je_i}A_{Je_a}e_b + A_{Je_b}A_{Je_i}A_{Je_j}e_a - A_{Je_b}A_{Je_j}A_{Je_i}e_a \\
&\quad + A_{Je_a}A_{Je_i}A_{Je_j}e_b - A_{Je_a}A_{Je_j}A_{Je_i}e_b.
\end{aligned}$$

Case (i): $1 = i \neq j$, $1 = a = b$:

$$\mu(\mu - \lambda)\,(\lambda - 2\mu)\,e_j - \lambda\mu^2 e_j + \lambda^2\mu e_j + \mu^3 e_j - \lambda\mu^2 e_j + \mu^3 e_j - \lambda\mu^2 e_j = 0.$$

Case (ii): $1 = i \neq j$, $1 = a \neq b$:

$$\delta_{bj}\left(\mu(\mu - \lambda)(\lambda - 2\mu)e_1 - \lambda\mu^2 e_1 + \mu^3 e_1 + \mu^3 e_1 - \lambda\mu^2 e_1 + \lambda^2\mu e_1 - \lambda\mu^2 e_1\right) = 0.$$

Case (iii): $1 = i \neq j$, $1 \neq a = b$:

$$\mu(\mu - \lambda)(\mu e_j + 2\delta_{aj}\mu e_a) - \mu^3 e_j + \lambda\mu^2 e_j + \delta_{aj}(\lambda\mu^2 e_a - \mu^3 e_a + \lambda\mu^2 e_a - \mu^3 e_a) = 0.$$

Case (iv): $1 = i \neq j$, $1 \neq a \neq b$:

$$\mu(\mu - \lambda)(\delta_{aj}\mu e_b + \delta_{bj}\mu e_a) + \delta_{aj}(\lambda\mu^2 e_b - \mu^2 e_b) + \delta_{bj}(\lambda\mu^2 e_a - \mu^2 e_a) = 0$$

Case (v): $1 \neq i \neq j$, $1 = a = b$:
every term vanishes on its own.

Case (vi): $1 \neq i \neq j$, $1 = a \neq b$:

$$-\delta_{bj}\mu^3 e_i + \delta_{bi}\mu^3 e_j + \delta_{bj}\mu^3 e_i - \delta_{bi}\mu^3 e_j = 0.$$

Case (vii): $1 \neq i \neq j$, $1 \neq a = b$:
every term vanishes on its own, using that $\delta_{ai}\delta_{aj} = 0$ since $i \neq j$.

Case (viii): $1 \neq i \neq j$, $1 \neq a \neq b$:

$$\delta_{aj}\delta_{bi}\mu^3 Je_1 - \delta_{ai}\delta_{bj}\mu^3 Je_1 + \delta_{ai}\delta_{bj}\mu^3 Je_1 - \delta_{aj}\delta_{bi}\mu^3 Je_1 = 0.$$

All 8 cases together prove that M has pseudo-parallel cubic form. □

Proposition 3.5.10. *Let M be a Lagrangian submanifold with pseudo-parallel cubic form for a function ϕ. Then M is H-pseudo-parallel for that same ϕ.*

Proof. Consider the condition for having pseudo-parallel cubic form, written in terms of h and the Lagrangian wedge:

$$(\bar{R} - \phi\bar{\wedge}) \cdot h \equiv 0.$$

Expanding this, we get

$$\begin{aligned} 0 &= R^{\perp}(X,Y)h(U,V) - h(R(X,Y)U,V) - h(U,R(X,Y)V) \\ &\quad - \phi(JX \wedge JY)h(U,V) + \phi h((X \wedge Y)U,V) + \phi h(U,(X \wedge Y)V). \end{aligned}$$

Let $\{e_1, \ldots, e_n\}$ be any orthonormal basis of T_pM. Then take $U = V = e_i$ to obtain

$$\begin{aligned} 0 &= \frac{1}{n}\sum_{i=1}^{n} \big(R^{\perp}(X,Y)h(e_i,e_i) - 2h(R(X,Y)e_i,e_i) \\ &\qquad -\phi(JX \wedge JY)h(e_i,e_i) + \phi 2h((X \wedge Y)e_i,e_i)\big) \\ &= R^{\perp}(X,Y)H - \phi(JX \wedge JY)H \\ &\qquad - \frac{2}{n}\sum_{i=1}^{n} \left(h(R(X,Y)e_i,e_i) - \phi h((X \wedge Y)e_i,e_i)\right). \end{aligned}$$

It now suffices to prove the last two terms drop for all $i \in \{1, \ldots, n\}$. Let $Z \in T_pM$, we may assume that the basis $\{e_1, \ldots, e_n\}$ diagonalises A_{JZ}, i.e. $h(Z,e_i) = \lambda_i Je_i$ for certain λ_i. So take the inner product of the last two terms with JZ:

$$\begin{aligned} &\langle h(R(X,Y)e_i,e_i), JZ\rangle - \phi\langle h((X \wedge Y)e_i,e_i), JZ\rangle \\ &= \langle h(Z,e_i), JR(X,Y)e_i\rangle - \phi\langle h(Z,e_i), J(X \wedge Y)e_i\rangle \\ &= -\lambda_i\left(\langle e_i, R(X,Y)e_i\rangle - \phi\langle e_i, (X \wedge Y)e_i\rangle\right) = 0, \end{aligned}$$

because of the symmetries of curvature tensors. So M is H-pseudo-parallel for ϕ. □

We can give an analogue of theorem 3.2.8 where we weaken the condition of minimality to pseudo-umbilicity and obtain a slightly weaker result in return:

Theorem 3.5.11. *Let $M^n(c)$ be a pseudo-umbilical Lagrangian submanifold of constant sectional curvature c immersed in a complex space form $\tilde{M}^n(4\tilde{c})$. Then $c = 0$ or $c = \tilde{c}$.*

Proof. Let $p \in M$. If $h_p \equiv 0$, then M has constant sectional curvature $\tilde{c}$ at p and thus $c = \tilde{c}$. So we assume $h_p \not\equiv 0$. On the one hand, because M has constant curvature c, it has pseudo-parallel cubic form at p for the function $\phi(p) = c$. But then it is also H-pseudo-parallel for the function $\phi(p) = c$. On the other hand, because M is pseudo-umbilical, by theorem 3.3.17 it is H-pseudo-parallel at p for the function $\psi(p) = \tilde{c}$.

Now, we have two possible cases: $H(p) = 0$ or $H(p) \neq 0$. In the first case, we find by theorem 3.2.8 that $c = 0$. In the second case, we find that $\phi(p) = \psi(p)$ by theorem 3.2.10, or thus $c = \tilde{c}$. □

For Lagrangian surfaces we obtain the following:

Proposition 3.5.12. *Let M^2 be a Lagrangian surface. Then M has pseudo-parallel cubic form for the function $\phi = K$ with K the Gaussian curvature.*

Proof. A surface always has curvature tensor $R = K\wedge$. So $(R - K\wedge) \equiv 0$ and thus M has pseudo-parallel cubic form for $\phi = K$. □

We have already remarked that the conditions of pseudo-parallelity and having pseudo-parallel cubic form are different. In fact, we can show the former is strictly stronger than the latter.

Proposition 3.5.13. *If M is pseudo-parallel, then it has pseudo-parallel cubic form.*

Proof. We first consider the case $n = 2$. All 2-dimensional Lagrangian submanifolds have pseudo-parallel cubic form by taking $\phi = K$, the Gaussian curvature. Thus the implication clearly holds. For the case $n \geq 3$, note that pseudo-parallel submanifolds are semi-parallel and therefore have pseudo-parallel cubic form for $\phi = 0$. □

3.6 Summary

For a Lagrangian surface of a complex space form, we give the following graphical summary of the constraints mentioned in this chapter:

$$
\begin{tikzcd}
 & & \text{Lagr.-umb.} \arrow[dll, Rightarrow, bend right] & \\
\text{Ps.-umb.} \arrow[d, Rightarrow] & H=0 \arrow[l, Rightarrow] \arrow[d, Rightarrow] & h\equiv 0 \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[u, Rightarrow] \arrow[d, Rightarrow] & K=\tilde{c} \arrow[d, Rightarrow] \\
H\text{-umb.} & \nabla^{\perp}H\equiv 0 \arrow[d, Rightarrow] & \bar{\nabla}h\equiv 0 \arrow[l, Rightarrow] \arrow[d, Rightarrow] & K=c \\
 & R^{\perp}H\equiv 0 & \bar{R}\cdot h\equiv 0 \arrow[l, Rightarrow] & K=0 \arrow[l, Rightarrow] \arrow[u, Rightarrow]
\end{tikzcd}
$$

We left out the condition of pseudo-parallelity as it is always equivalent to H-semi-parallelity, as shown in corollary 3.4.9.

For a Lagrangian submanifold of a complex space form of arbitrary dimension $n \geq 3$, we obtain the following summary:

$$
\begin{tikzcd}
 & & \text{Lagr.-umb.} \arrow[dll, Rightarrow, bend right] \arrow[r, Rightarrow] & H\text{-umb.} \arrow[ddddl, Rightarrow, bend left] \\
\text{Ps.-umb.} \arrow[dddr, Rightarrow] & H=0 \arrow[l, Rightarrow] \arrow[d, Rightarrow] & h\equiv 0 \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[u, Rightarrow] \arrow[d, Rightarrow] & K=\tilde{c} \arrow[ddd, Rightarrow, bend left] \\
 & \nabla^{\perp}H\equiv 0 \arrow[d, Rightarrow] & \bar{\nabla}h\equiv 0 \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \\
 & R^{\perp}H\equiv 0 \arrow[d, Rightarrow] & \bar{R}\cdot h\equiv 0 \arrow[l, Rightarrow] \arrow[d, Rightarrow] & K=0 \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
 & (R^{\perp}-\phi\wedge^{\perp})H\equiv 0 & (R-\phi\wedge)\cdot C\equiv 0 \arrow[l, Rightarrow] & K=c \arrow[l, Rightarrow]
\end{tikzcd}
$$

We left out the condition of pseudo-parallelity as it is always equivalent to semi-parallelity, as shown in theorem 3.4.7.

A last remark in this chapter, we will give examples of Lagrangian manifolds that show that the conditions of pseudo-parallel cubic form and H-pseudo-parallelity are not equivalent to any of the stronger conditions in the summary above.

We begin with the condition of pseudo-parallel cubic form. Theorem 3.5.9 tells us that H-umbilical Lagrangian submanifolds have pseudo-parallel cubic form. An example is the Whitney sphere given in example 3.3.11, which does not have constant sectional curvature. Now consider the following examples:

Example 3.6.1. In [Toj01], Tojeiro gives explicit examples of non-totally geodesic Lagrangian submanifolds $M^n(\tilde{c})$ with constant sectional curvature $\tilde{c}$ immersed in a complex space form $\tilde{M}^n(4\tilde{c})$ with $\tilde{c} \neq 0$. So these submanifolds have pseudo-parallel cubic form for the function $\phi = \tilde{c} \neq 0$. Thus they are not semi-parallel.

Example 3.6.2. In [Mat94; Xia92], it is proven that there exist standard embeddings of $SU(3)/SO(3)$ $(n = 5)$; $SU(6)/Sp(3)$ $(n = 14)$, $SU(3)$ $(n = 8)$ and E_6/F_4 $(n = 26)$ in $\mathbb{C}P^n(4\tilde{c})$ such that for any unit tangent vector X:

$$\|h(X, X)\| = \sqrt{\tilde{c}/2}.$$

Thus there can be no orthonormal basis of the tangent space satisfying the requirements for H-umbilicity. However, these embeddings have parallel second fundamental form and thus have pseudo-parallel cubic form.

Indeed, we find that the condition of pseudo-parallel cubic form is not equivalent to semi-parallellity, H-umbilicity or having constant sectional curvature.

We move on to the condition of H-pseudo-parallellity. Combining proposition 3.5.10 and theorem 3.5.9, we have that a H-umbilical Lagrangian submanifold is H-pseudo-parallel. Again, we have the Whitney sphere given in example 3.3.11 which has $\lambda = 3\mu$ and is thus not pseudo-umbilical. From example 3.6.1 we have non-totally geodesic Lagrangian submanifolds of constant sectional curvature $\tilde{c} \neq 0$. By corollary 3.2.9 these submanifolds are not H-semi-parallel. Finally, we have the following example:

Example 3.6.3. In [CPM12] examples are given of Lagrangian submanifolds in $\mathbb{C}^5$ for which there exists an orthonormal frame $\{e_1, \ldots, e_5\}$ such that the cubic form takes the following form:

$$\begin{matrix} C_{111} = \lambda & C_{122} = -a \\ C_{333} = \mu & C_{344} = -b \end{matrix} \qquad C_{ijk} = 0 \text{ otherwise},$$

where λ, μ are nonzero functions. In particular, these examples are minimal and thus H-pseudo-parallel. However, we can show they do not have pseudo-parallel cubic form. Assume a Lagrangian submanifold M^5 has this second fundamental form. Consider the condition

$$(R(X, Y) - \phi(X \wedge Y)) \cdot C(U, V, W),$$

then choosing $X = e_1, Y = U = V = W = e_2$ gives (after a short, simple calculation) that

$$\phi = \tilde{c} - 2\lambda^2.$$

On the other hand, choosing $X = U = e_3, Y = V = W = e_1$ gives (after an even shorter and simpler calculation) that

$$\phi = \tilde{c}.$$

However, since $\lambda \neq 0$, ϕ is unique and these two results do not match. Thus M does not have pseudo-parallel cubic form, since the required function ϕ does not exist.

Chapter 4

Decomposition of the tangent space of a Lagrangian submanifold

When attempting to classify certain Lagrangian submanifolds, it is often vital to obtain a suitable basis for the tangent space, or a useful decomposition into vector subspaces. In this chapter, we will assume certain constraints on a Lagrangian submanifold M^n of a complex space form $\tilde{M}^n(4\tilde{c})$, and attempt to give such decompositions. We will use a canonical basis and techniques similar to those applied in [Dil+12; Eji82].

Throughout this chapter, we assume that $\{e_1, \ldots, e_n\}$ is a canonical basis and $\{\lambda_1, \ldots, \lambda_n\}$ their eigenvalues. We define the following important functions:

$$\nu = \frac{1}{2}\lambda_1 \geq 0,$$
$$\eta = \frac{1}{2}\sqrt{\lambda_1^2 + 4(\tilde{c} - \phi)} = \sqrt{\nu^2 + \tilde{c} - \phi} \geq 0.$$

Then we have that $\tilde{c} - \phi = \eta^2 - \nu^2$ and $\lambda_1 = 2\nu$. Moreover, we will always assume that $\nu > 0$, since otherwise $\lambda_1 = 0$ and M is then totally geodesic.

4.1 H-pseudo-parallel

We will start with the weakest condition of all, that of H-pseudo-parallelity:

$$R(X,Y)JH - \phi(X \wedge Y)JH = 0.$$

When we rewrite this using the equation of Gauss (2.1.7) and take the inner product with a tangent vector Z, we get

$$(\eta^2 - \nu^2)\langle (X \wedge Y)JH, Z\rangle + \langle [A_{JX}, A_{JX}]JH, Z\rangle = 0.$$

Let us take $X = e_a, Y = e_b, Z = e_c$ with $a \neq b$. Then we find

$$\begin{aligned}
0 &= (\eta^2 - \nu^2)\langle (e_a \wedge e_b) JH, e_c\rangle + \langle [A_{Je_a}, A_{Je_b}] JH, e_c\rangle \\
&= -(\eta^2 - \nu^2)\langle (e_a \wedge e_b) e_c, JH\rangle - \langle [A_{Je_a}, A_{Je_b}] e_c, JH\rangle \\
&= (\eta^2 - \nu^2)(\delta_{ac}\langle e_b, JH\rangle - \delta_{bc}\langle e_a, JH\rangle) + \sum_{i=1}^{n} \langle e_i, JH\rangle \left(\langle A_{Je_b} A_{Je_a} e_c, e_i\rangle - \langle A_{Je_a} A_{Je_b} e_c, e_i\rangle\right) \\
&= (\eta^2 - \nu^2)(\delta_{ac}\langle e_b, JH\rangle - \delta_{bc}\langle e_a, JH\rangle) + \sum_{i=1}^{n} \langle e_i, JH\rangle \left(\langle A_{Je_a} e_c, A_{Je_b} e_i\rangle - \langle A_{Je_b} e_c, A_{Je_a} e_i\rangle\right) \\
&= (\eta^2 - \nu^2)(\delta_{ac}\langle e_b, JH\rangle - \delta_{bc}\langle e_a, JH\rangle) + \sum_{i=1}^{n} \langle e_i, JH\rangle \sum_{j=1}^{n} \left(C_{acj} C_{bij} - C_{bcj} C_{aij}\right). \quad (4.1.1)
\end{aligned}$$

Lemma 4.1.1. *If $\langle H, Je_i\rangle \neq 0$ for $2 \leq i \leq n$, then $\lambda_i = \nu - \eta$.*

Proof. Take $a = c = 1 \neq b$ in (4.1.1). Then we find

$$\begin{aligned}
0 &= (\eta^2 - \nu^2)\langle e_b, JH\rangle + \sum_{i=1}^{n} \langle e_i, JH\rangle \sum_{j=1}^{n} \left(C_{11j} C_{bij} - C_{b1j} C_{1ij}\right) \\
&= \langle e_b, JH\rangle \left(\eta^2 - \nu^2 + 2\nu\lambda_b - \lambda_b^2\right) \\
&= \langle e_b, JH\rangle (\lambda_b - \nu + \eta)(\lambda_b - \nu - \eta).
\end{aligned}$$

We know that $\lambda_b \leq \nu$ because of the properties of the canonical basis, thus the third factor is nonzero. So if $\langle e_b, JH\rangle \neq 0$, then $\lambda_b = \nu - \eta$. □

Corollary 4.1.2. *Suppose that W is the eigenspace of $\nu - \eta$, i.e. $W = \operatorname{span}\{e_m, \ldots, e_n\}$ with $m \geq 2$, and U is the (1-dimensional) eigenspace of 2ν. Then $JH \in U \oplus W$.*

Lemma 4.1.3. *If $W = \emptyset$, i.e. none of the vectors $\{e_2, \ldots, e_n\}$ have eigenvalue $\nu - \eta$, then M is minimal.*

Proof. If $W = \emptyset$, then $JH \in U$, or thus $JH = \langle JH, e_1\rangle e_1$. Consider $1 = a \neq b = c$ in (4.1.1):

$$\begin{aligned}
0 &= -(\eta^2 - \nu^2)\langle e_1, JH\rangle + \sum_{i=1}^{n} \langle e_i, JH\rangle \sum_{j=1}^{n} \left(C_{1bj} C_{bij} - C_{bbj} C_{1ij}\right) \\
&= -(\eta^2 - \nu^2)\langle e_1, JH\rangle + \langle e_1, JH\rangle \sum_{j=1}^{n} \left(C_{1bj} C_{b1j} - C_{bbj} C_{11j}\right) \\
&= \langle e_1, JH\rangle \left(\lambda_b^2 - 2\nu\lambda_b - \eta^2 + \nu^2\right) \\
&= \langle e_1, JH\rangle (\lambda_b - \nu - \eta)(\lambda_b - \nu + \eta).
\end{aligned}$$

The last two factors must now be nonzero, so $\langle e_1, JH\rangle = 0$ and therefore M is minimal. □

Theorem 4.1.4. *Let M be a H-pseudo-parallel Lagrangian submanifold. Then one of the following two situations happens:*

(i) M is minimal,

(ii) we can decompose the tangent space in 3 *mutually orthogonal subspaces* $T_pM = U \oplus V \oplus W$ *such that*

$$\begin{array}{ll} h(U,U) \subset JU & h(U,W) \subset JW \\ h(U,V) \subset JV & h(V,W) \subset JV \oplus JW, \end{array}$$

where U *is* 1*-dimensional and* U, W *are always nonempty. Moreover,* $H \in JU \oplus JW$*.*

Proof. Assume M is not minimal. We define U to be the eigenspace of 2ν and W the eigenspace of $\nu - \eta$. Then $U = \operatorname{span}\{e_1\}$ and is therefore 1-dimensional and nonempty. The subspace W is nonempty because of lemma 4.1.3. The four listed properties follow from the properties of the canonical basis. Finally, $H \in JU \oplus JW$ due to (4.1.2). □

4.2 Pseudo-parallel cubic form

This section is based on section 3 of [Dil+12], but generalised from semi-parallelity to pseudo-parallel cubic form. The same decomposition was used there to classify parallel Lagrangian submanifolds of $\mathbb{C}P^n$.

Using equation (2.1.7), the condition for pseudo-parallel cubic form becomes

$$\underset{U,V,W}{\Xi} \left\{(\eta^2 - \nu^2) C((X \wedge Y)U, V, W) + C([A_{JX}, A_{JY}]U, V, W)\right\} = 0.$$

We expand the wedge in the first term to get

$$(\eta^2 - \nu^2)(\langle Y, U\rangle C(X, V, W) - \langle X, U\rangle C(Y, V, W)).$$

For the second term, we expand the Lie bracket and then apply the following steps:

$$\begin{aligned}
&C(A_{JX}A_{JY}U, V, W) \\
&= \langle JA_{JX}A_{JY}U, h(V,W)\rangle = -\langle A_{JX}A_{JY}U, Jh(V,W)\rangle = -\langle h(Y,U), h(X, Jh(V,W))\rangle \\
&= C(Y, U, Jh(X, Jh(V,W))) = C\left(Y, U, \sum_{i=1}^{n} \langle Jh(X, Jh(V,W)), e_i\rangle e_i\right) \\
&= -\sum_{i=1}^{n} C\left(Y, U, e_i\right) C(X, Jh(V,W), e_i) \\
&= -\sum_{i=1}^{n} C\left(Y, U, e_i\right) C\left(X, \sum_{j=1}^{n} \langle Jh(V,W), e_j\rangle e_j, e_i\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} C\left(Y, U, e_i\right) C\left(X, e_i, e_j\right) C(V, W, e_j),
\end{aligned}$$

and similarly

$$C(A_{JY}A_{JX}U, V, W) = \sum_{i=1}^{n}\sum_{j=1}^{n} C\left(X, U, e_i\right) C\left(Y, e_i, e_j\right) C(V, W, e_j).$$

The second term then becomes

$$\sum_{j=1}^{n} C(V,W,e_j)\sum_{i=1}^{n}\left(\left(C\left(Y,U,e_i\right)C\left(X,e_i,e_j\right)-C\left(X,U,e_i\right)C\left(Y,e_i,e_j\right)\right)\right).$$

So putting everything together, we find that

$$\begin{aligned}0=\underset{U,V,W}{\Xi}\Bigg\{&(\eta^2-\nu^2)(\langle Y,U\rangle C(X,V,W)-\langle X,U\rangle C(Y,V,W)) \\ &+\sum_{j=1}^{n} C(V,W,e_j)\sum_{i=1}^{n}\left(C\left(Y,U,e_i\right)C\left(X,e_i,e_j\right)-C\left(X,U,e_i\right)C\left(Y,e_i,e_j\right)\right)\Bigg\}.\end{aligned} \tag{4.2.1}$$

Let's assume $X=e_a, Y=e_b, U=e_c, V=e_d, W=e_e$. Then we obtain

$$0=\underset{c,d,e}{\Xi}\left\{(\eta^2-\nu^2)(\delta_{bc}C_{ade}-\delta_{ac}C_{bde})+\sum_{j=1}^{n} C_{dej}\sum_{i=1}^{n}(C_{bci}C_{aij}-C_{aci}C_{bij})\right\}. \tag{4.2.2}$$

Lemma 4.2.1. *The tangent space T_pM can be decomposed as a direct sum of 3 orthogonal vector spaces, that is, $T_pM=U\oplus V\oplus W$ where*

- *(i) U is a 1-dimensional vector space spanned by e_1,*
- *(ii) $h(e_1,e_1)=2\nu Je_1$,*
- *(iii) $h(e_1,v)=\nu Jv$ for any $v\in V$,*
- *(iv) $h(e_1,w)=(\nu-\eta)Jw$ for any $w\in W$,*
- *(v) $h(v_1,v_2)-\nu\langle v_1,v_2\rangle Je_1\in JW$ for any $v_1,v_2\in V$,*
- *(vi) $h(w_1,w_2)-(\nu-\eta)\langle w_1,w_2\rangle Je_1\in JW$ for any $w_1,w_2\in W$,*
- *(vii) $h(v,w)\in JV$ for all $v\in V$, $w\in W$.*

Proof. Items (i) and (ii) follow directly from the definition of the canonical basis. Choose $a=c=d=1$ and $b=e\neq 1$. Then (4.2.2) gives

$$\begin{aligned}0&=2\left(-(\eta^2-\nu^2)C_{1bb}+\sum_{j=1}^{n} C_{1bj}\sum_{i=1}^{n}(C_{1bi}C_{1ij}-C_{11i}C_{bij})\right)\\ &\quad+(\eta^2-\nu^2)C_{111}+\sum_{j=1}^{n} C_{11j}\sum_{i=1}^{n}(C_{bbi}C_{1ij}-C_{1bi}C_{bij})\\ &=-2(\eta^2-\nu^2)\lambda_b+2\lambda_b^3-4\nu\lambda_b^2+2(\eta^2-\nu^2)\nu+4\nu^2\lambda_b-\nu\lambda_b^2\\ &=(\lambda_b-\nu)(\lambda_b-\nu-\eta)(\lambda_b-\nu+\eta).\end{aligned}$$

Thus for $b\in\{2,\ldots,n\}$, there are only two possible values for λ_b: ν and $\nu-\eta$. Let V be the eigenspace of the eigenvalue ν and W the eigenspace of the eigenvalue $\nu-\eta$. By linearity, items (iii) and (iv) are then proven.

For items (v) and (vi), let $a = c = 1$ and b, d, e different from 1.

$$\begin{aligned}
0 = & -(\eta^2 - \nu^2) C_{bde} + \sum_{j=1}^{n} C_{dej} \sum_{i=1}^{n} (C_{1bi}C_{1ij} - C_{11i}C_{bij}) \\
& + (\eta^2 - \nu^2)\delta_{bd} C_{11e} + \sum_{j=1}^{n} C_{1ej} \sum_{i=1}^{n} (C_{bdi}C_{1ij} - C_{1di}C_{bij}) \\
& + (\eta^2 - \nu^2)\delta_{be} C_{11d} + \sum_{j=1}^{n} C_{1dj} \sum_{i=1}^{n} (C_{bei}C_{1ij} - C_{1ei}C_{bij}) \\
= & \, C_{bde} \left(-\eta^2 + \nu^2 + \lambda_b^2 - 2\nu\lambda_b + \lambda_e^2 - \lambda_d\lambda_e + \lambda_d^2 - \lambda_d\lambda_e \right) \\
= & \, C_{bde} \left((\lambda_b - \nu - \eta)(\lambda_b - \nu + \eta) + (\lambda_d - \lambda_e)^2 \right).
\end{aligned}$$

If $\lambda_b = \lambda_d = \lambda_e = \nu$, then we find that $C_{bde} = 0$. So by linearity $C(v_1, v_2, v_3) = 0$ for all $v_1, v_2, v_3 \in V$. So $h(V, V) \subset JU \oplus JW$. By item (iii) and the symmetry of the cubic form, we know that

$$\langle h(v_1, v_2), Je_1 \rangle = \nu \langle v_1, v_2 \rangle,$$

and thus $h(v_1, v_2) - \nu\langle v_1, v_2\rangle Je_1 \in JW$, which proves item (v).

If $\lambda_b = \nu$ and $\lambda_d = \lambda_e = \nu - \eta$, then again $C_{bde} = 0$. So by linearity, $C(w_1, w_2, v) = 0$ for all $v \in V$, $w_1, w_2 \in W$. So $h(W, W) \subset JU \oplus JW$. By item (iv) and the symmetry of the cubic form, we know that

$$\langle h(w_1, w_2), Je_1 \rangle = (\nu - \eta)\langle w_1, w_2 \rangle,$$

and thus $h(w_1, w_2) - (\nu - \eta)\langle w_1, w_2\rangle Je_1 \in JW$, which proves item (vi) .

For item (vii), note that $\langle h(v, w), Je_1 \rangle = 0$ for all $v \in V$ and $w \in W$, and recall that we have proven that

$$C_{abc} \left((\lambda_a - \nu - \eta)(\lambda_a - \nu + \eta) + (\lambda_b - \lambda_c)^2 \right) = 0,$$

for $a, b, c \neq 1$. Choosing $2 \leq a \leq m$ and $m + 1 \leq b, c \leq n$ gives us that $C_{abc} = 0$, and thus by linearity $C(v, w, w') = 0$ for all $v \in V$ and $w, w' \in W$. So for all $v \in V$ and $w \in W$, we find that $h(v, w) \in JV$. □

We have n possible cases for the eigenvectors:
Case 1: $\lambda_2 = \cdots = \lambda_n = \nu$.
Case m: $\lambda_2 = \cdots = \lambda_m = \nu$ and $\lambda_{m+1} = \ldots = \lambda_n = \nu - \eta$ for $2 \leq m \leq n - 1$.
Case n: $\lambda_2 = \cdots = \lambda_n = (\nu - \eta)$.

If $\eta = 0$, all cases are identical. We will simply consider that a part of case n, and from now on assume that $\eta \neq 0$.

Proposition 4.2.2. *Case 1 does not occur.*

Proof. Assume we are in case 1. Let $a = 1$ and $b = c = d = e \neq 1$. Then

$$\begin{aligned}
0 &= (\eta^2 - \nu^2)C_{1bb} + \sum_{j=1}^{n} C_{bbj} \sum_{i=1}^{n} (C_{bbi}C_{1ij} - C_{1bi}C_{bij}) \\
&= (\eta^2 - \nu^2)\lambda_b + \sum_{j=1}^{n} C_{bbj}^2(\lambda_j - \lambda_b) \\
&= (\eta^2 - \nu^2)\lambda_b + C_{bb1}^2(2\nu - \lambda_b) + \sum_{j=2}^{n} C_{bbj}^2(\lambda_j - \lambda_b) \\
&= -\lambda_b(\lambda_b - \nu - \eta)(\lambda_b - \nu + \eta) + \sum_{j=2}^{n} C_{bbj}^2(\lambda_j - \lambda_b).
\end{aligned}$$

Since $\lambda_2 = \cdots = \lambda_n = \lambda_b = \nu$, then this reduces to

$$0 = \nu\eta^2,$$

but as we assumed $\nu \neq 0$ and $\eta \neq 0$ this gives a contradiction. □

We will now work in Case m with $2 \leq m \leq n + 1$.

Definition 4.2.3. We introduce a bilinear map $L : V \times V \to W$ by

$$L(v_1, v_2) := -J(h(v_1, v_2) - \nu\langle v_1, v_2\rangle Je_1),$$

which indeed has image in W due to property (v) in lemma 4.2.1.

Then clearly we have

$$L(v_1, v_2) = \sum_{i=m+1}^{n} C(v_1, v_2, e_i)e_i,$$

or for vectors of the canonical basis,

$$L(e_j, e_k) = \sum_{i=m+1}^{n} C_{ijk}e_i,$$

and thus

$$\langle L(e_j, e_k), L(e_r, e_s)\rangle = \sum_{i=m+1}^{n} C_{ijk}C_{irs}. \tag{4.2.3}$$

Lemma 4.2.4. *The tensor L is an isotropic tensor, i.e.*

$$\|L(v, v)\|^2 = \nu\eta,$$

for a unit vector $v \in V$. Moreover, for $v_1, v_2, v_3, v_4 \in V$ unit vectors, we have that

$$\begin{aligned}
&\langle L(v_1, v_2), L(v_3, v_4)\rangle + \langle L(v_1, v_3), L(v_2, v_4)\rangle + \langle L(v_1, v_4), L(v_2, v_3)\rangle \\
&= \nu\eta \left(\langle v_1, v_2\rangle\langle v_3, v_4\rangle + \langle v_1, v_3\rangle\langle v_2, v_4\rangle + \langle v_1, v_4\rangle\langle v_2, v_3\rangle\right).
\end{aligned}$$

Proof. By linearity, it suffices to prove this for the vectors of the canonical basis that span V. Let $a = 1$ and $2 \leq b, c, d, e \leq m$. Then

$$\begin{aligned}
0 &= (\eta^2 - \nu^2)\delta_{bc}C_{1de} + \sum_{j=1}^{n} C_{dej} \sum_{i=1}^{n} (C_{bci}C_{1ij} - C_{1ci}C_{bij}) \\
&\quad + (\eta^2 - \nu^2)\delta_{bd}C_{1ce} + \sum_{j=1}^{n} C_{cej} \sum_{i=1}^{n} (C_{bdi}C_{1ij} - C_{1di}C_{bij}) \\
&\quad + (\eta^2 - \nu^2)\delta_{be}C_{1cd} + \sum_{j=1}^{n} C_{cdj} \sum_{i=1}^{n} (C_{bei}C_{1ij} - C_{1ei}C_{bij}) \\
&= (\eta^2 - \nu^2)\nu(\delta_{bc}\delta_{de} + \delta_{bd}\delta_{ce} + \delta_{be}\delta_{cd}) + \sum_{j=1}^{n} (\lambda_j - \nu)\,(C_{dej}C_{bcj} + C_{cej}C_{bdj} + C_{cdj}C_{bej}) \\
&= ((\eta^2 - \nu^2)\nu + \nu^3)(\delta_{bc}\delta_{de} + \delta_{bd}\delta_{ce} + \delta_{be}\delta_{cd}) - \eta \sum_{j=m+1}^{n} (C_{dej}C_{bcj} + C_{cej}C_{bdj} + C_{cdj}C_{bej}) \\
&= \eta \left(\nu\eta(\delta_{bc}\delta_{de} + \delta_{bd}\delta_{ce} + \delta_{be}\delta_{cd}) - \sum_{j=m+1}^{n} C_{dej}C_{bcj} - \sum_{j=m+1}^{n} C_{cej}C_{bdj} - \sum_{j=m+1}^{n} C_{cdj}C_{bej} \right).
\end{aligned}$$

Due to $\eta \neq 0$ and (4.2.3), this proves the second part of the theorem. In particular, if v is a unit vector, then taking $v = v_1 = v_2 = v_3 = v_4$ in the second part of the theorem gives

$$\|L(v, v)\|^2 = \nu\eta,$$

so L is isotropic. □

We now decompose W as the direct sum of two orthogonal vector spaces: $W_1 = L(V, V)$ and its orthogonal complement in W, named W_2. By definition of L, we have that $h(V, V) \in JU \oplus JW_1$.

Next, we give a characterisation of W_2:

Lemma 4.2.5. *$w \in W_2$ if and only if $h(v, w) = 0$ $\forall v \in V$.*

Proof. By property (vii) in lemma 4.2.1, we need only consider the JV-component. For any $v_1, v_2 \in V$ and $w \in W$, we have

$$\langle h(v_1, w), Jv_2 \rangle = \langle h(v_1, v_2), Jw \rangle - \nu \langle v_1, v_2 \rangle \langle Je_1, Jw \rangle = \langle L(v_1, v_2), w \rangle,$$

and the lemma follows easily from this equality. □

From this we also see that $h(W_1, W_2) \subset JW$. We can improve this to $h(W_1, W_2) \subset JW_2$:

Lemma 4.2.6. *Let $v_1, v_2 \in V$ and $w \in W_2$. Then*

$$h(L(v_1, v_2), w) = (\nu - \eta)\eta \langle v_1, v_2 \rangle Jw.$$

Proof. Again, by linearity, it suffices to prove this for vectors of the canonical basis. Choose $a = 1$, $2 \leq b, c \leq m$, $m+2 \leq d, e \leq n$ such that $e_d \in W$ and $e_e \in W_2$. Then

$$\begin{aligned}
0 &= (\eta^2 - \nu^2)\delta_{bc}C_{1de} + \sum_{j=1}^{n} C_{dej} \sum_{i=1}^{n} (C_{bci}C_{1ij} - C_{1ci}C_{bij}) \\
&\quad + \sum_{j=1}^{n} C_{cej} \sum_{i=1}^{n} (C_{bdi}C_{1ij} - C_{1di}C_{bij}) + \sum_{j=1}^{n} C_{cdj} \sum_{i=1}^{n} (C_{bei}C_{1ij} - C_{1ei}C_{bij}) \\
&= (\eta^2 - \nu^2)(\nu - \eta)\delta_{bc}\delta_{de} + \sum_{j=1}^{n} (\lambda_j - \nu)C_{bcj}C_{dej} \\
&= \eta^2(\nu - \eta)\delta_{bc}\delta_{de} - \eta \sum_{j=m+1}^{n} C_{bcj}C_{dej} \\
&= \eta\left(\eta(\nu - \eta)\delta_{bc}\delta_{de} - \langle JL(e_b, e_c), h(e_d, e_e)\rangle\right) \\
&= \eta\left(\eta(\nu - \eta)\delta_{bc}\langle Je_d, Je_e\rangle - \langle Je_d, h(L(e_b, e_c), e_e)\rangle\right),
\end{aligned}$$

and since $\eta \neq 0$ we find

$$h(L(e_b, e_c), e_e) = \eta(\nu - \eta)\delta_{bc}Je_e,$$

which proves the lemma. □

Finally, we show that $h(W_1, W_1) \in JU \oplus JW_1$:

Lemma 4.2.7. *Let $v_1, v_2, v_3, v_4 \in V$, then we have that*

$$\begin{aligned}
&h(L(v_1, v_2), L(v_3, v_4)) \\
&\quad = (\nu - \eta)\langle L(v_1, v_2), L(v_3, v_4)\rangle Je_1 + (\nu - \eta)\eta\langle v_1, v_2\rangle JL(v_3, v_4) \\
&\qquad + \sum_{i=2}^{m} \langle L(v_1, e_i), L(v_3, v_4)\rangle JL(v_2, e_i) + \sum_{i=2}^{m} \langle L(v_2, e_i), L(v_3, v_4)\rangle JL(v_1, e_i).
\end{aligned}$$

Proof. Proving the Je_1-component can be done by taking the inner product with Je_1:

$$\langle h(L(v_1, v_2), L(v_3, v_4)), Je_1\rangle = (\nu - \eta)\langle L(v_1, v_2), L(v_3, v_4)\rangle.$$

Now suppose $a = 1$, $2 \leq b, c \leq m$, $m+1 \leq d, e \leq n$ such that $e_d \in W_1$.

$$
\begin{aligned}
0 &= (\eta^2 - \nu^2)\delta_{bc} C_{1de} + \sum_{j=1}^{n} C_{dej} \sum_{i=1}^{n} (C_{bci} C_{1ij} - C_{1ci} C_{bij}) \\
&\quad + \sum_{j=1}^{n} C_{cej} \sum_{i=1}^{n} (C_{bdi} C_{1ij} - C_{1di} C_{bij}) + \sum_{j=1}^{n} C_{cdj} \sum_{i=1}^{n} (C_{bei} C_{1ij} - C_{1ei} C_{bij}) \\
&= (\eta^2 - \nu^2)(\nu - \eta)\delta_{bc}\delta_{de} + \sum_{j=1}^{n} (\lambda_j - \nu) \left(C_{dej} C_{bcj} + C_{cej} C_{bdj} + C_{cdj} C_{bej} \right) \\
&= \eta^2 (\nu - \eta)\delta_{bc}\delta_{de} - \eta \sum_{j=m+1}^{n} C_{dej} C_{bcj} + \eta \sum_{j=2}^{m} (C_{cej} C_{bdj} + C_{cdj} C_{bej}) \\
&= \eta^2 (\nu - \eta)\delta_{bc} \langle J e_d, J e_e \rangle - \eta \langle h(L(e_b, e_c), e_d), J e_e \rangle \\
&\quad + \eta \sum_{j=2}^{m} \langle L(e_b, e_j), e_d \rangle \langle J L(e_c, e_j), J e_e \rangle + \eta \sum_{j=2}^{m} \langle L(e_c, e_j), e_d \rangle \langle J L(e_b, e_j), J e_e \rangle .
\end{aligned}
$$

Since $\eta \neq 0$, we obtain

$$
\begin{aligned}
\langle h(L(e_b, e_c), e_d), J e_e \rangle = (\nu - \eta)\eta \delta_{bc} \langle J e_d, J e_e \rangle &+ \sum_{j=2}^{m} \langle L(e_b, e_j), e_d \rangle \langle J L(e_c, e_j), J e_e \rangle \\
&+ \eta \sum_{j=2}^{m} \langle L(e_c, e_j), e_d \rangle \langle J L(e_b, e_j), J e_e \rangle ,
\end{aligned}
$$

so by linearity, the lemma is proven. □

Theorem 4.2.8. *Suppose we are in Case m. Then we can decompose the tangent space $T_pM = U \oplus V \oplus W_1 \oplus W_2$ such that*

$$
\begin{array}{ll}
h(U,U) \subset JU & h(V,W_1) \subset JV \\
h(U,V) \subset JV & h(V,W_2) = 0 \\
h(U,W_1) \subset JW_1 & h(W_1,W_1) \subset JU \oplus JW_1 \\
h(U,W_2) \subset JW_2 & h(W_1,W_2) \subset JW_2 \\
h(V,V) \subset JU \oplus JW_1 & h(W_2,W_2) \subset JU \oplus JW_1 \oplus JW_2.
\end{array}
$$

Proof. The first 4 properties follow from (i) to (iv) in lemma 4.2.1. Moreover, the property $h(V,V) \subset JU \oplus JW_1$ follows property (v) in lemma 4.2.1 and the definition of W_1; $h(V,W_1) \subset JV$ follows from property (vii) in lemma 4.2.1; $h(V,W_2) = 0$ follows from lemma 4.2.5, $h(W_1,W_2) \subset JW_2$ follows from lemma 4.2.6 and $h(W_2,W_2) \subset JU \oplus JW_1 \oplus JW_2$ follows from lemma 4.2.7. □

Finally consider Case n. Then $\lambda_i = \nu - \eta$ for $i \in \{2, \ldots, n\}$. Since $e_2, \ldots, e_n$ are all in the eigenspace of $\nu - \eta$, we may assume that $e_2, \ldots, e_n$ is in fact the canonical basis of the vector space span$\{e_2, \ldots, e_n\}$:

$$
C_{2ij} = \lambda_i^{(2)} \delta_{ij},
$$

for $2 \leq i, j \leq n$. We define

$$\begin{aligned}
\nu_2 &= \frac{1}{2}\lambda_2^{(2)}, \\
\eta_2 &= \frac{1}{2}\sqrt{\lambda_2^{(2)^2} + 8\eta(\eta-\nu)} = \sqrt{\nu_2^2 + 2\eta(\eta-\nu)},
\end{aligned}$$

thus ${\eta_2}^2 - {\nu_2}^2 = 2\eta(\eta-\nu)$. So for $2 \leq a, b, c, d \leq n$ we have:

$$\begin{aligned}
0 = \underset{c,d,e}{\Xi} &\left\{ (\eta^2 - \nu^2)(\delta_{bc}C_{ade} - \delta_{ac}C_{bde}) + \sum_{j=2}^{n} C_{dej} \sum_{i=2}^{n} (C_{bci}C_{aij} - C_{aci}C_{bij}) \right. \\
&\left. + C_{1de} \sum_{i=1}^{n} (C_{bci}C_{1ai} - C_{aci}C_{1bi}) + \sum_{j=2}^{n} C_{dej} \left(C_{1bc}C_{1aj} - C_{1ac}C_{1bj}\right) \right\} \\
= \underset{c,d,e}{\Xi} &\left\{ (\eta^2 - \nu^2)(\delta_{bc}C_{ade} - \delta_{ac}C_{bde}) + \sum_{j=2}^{n} C_{dej} \sum_{i=2}^{n} (C_{bci}C_{aij} - C_{aci}C_{bij}) \right. \\
&\left. + (\nu - \eta)^2 \left(\delta_{bc}C_{ade} - \delta_{ac}C_{bde}\right) \right\} \\
= \underset{c,d,e}{\Xi} &\left\{ 2\eta(\eta-\nu)(\delta_{bc}C_{ade} - \delta_{ac}C_{bde}) + \sum_{j=2}^{n} C_{dej} \sum_{i=2}^{n} (C_{bci}C_{aij} - C_{aci}C_{bij}) \right\} \\
= \underset{c,d,e}{\Xi} &\left\{ (\eta_2^2 - \nu_2^2)(\delta_{bc}C_{ade} - \delta_{ac}C_{bde}) + \sum_{j=2}^{n} C_{dej} \sum_{i=2}^{n} (C_{bci}C_{aij} - C_{aci}C_{bij}) \right\}.
\end{aligned}$$

We can thus consider the vector subspace $\operatorname{span}\{e_2, \ldots, e_n\}$ as vectors satisfying the condition of pseudo-parallel cubic form again. We may then reapply the decomposition to this vector subspace.

4.3 Constant sectional curvature

This section is based on [Eji82], but is more general since we do not assume minimality. Suppose M is a manifold of constant sectional curvature c. Then the equation of Gauss (4.2.2) becomes

$$0 = (\eta^2 - \nu^2)\left(\langle Y, Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle\right) + \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle.$$

Lemma 4.3.1. *The eigenvalues of $\{e_2, \ldots, e_n\}$ are all $\nu - \eta$.*

Proof. If we choose $X = e_a, Y = e_b, Z = e_c, W = e_d$, then we find

$$0 = (\eta^2 - \nu^2)\left(\delta_{bc}\delta_{ad} - \delta_{ac}\delta_{bd}\right) + \sum_{i=1}^{n} \left(C_{bci}C_{adi} - C_{aci}C_{bdi}\right). \tag{4.3.1}$$

If we choose $a = c = 1 \neq b = d$ in (4.3.1), we get

$$\begin{aligned}
0 &= -(\eta^2 - \nu^2) + \sum_{i=1}^{n} (C_{b1i}C_{1bi} - C_{11i}C_{bbi}) \\
&= -(\eta^2 - \nu^2) + \lambda_b^2 - 2\nu\lambda_b \\
&= (\lambda_b - \nu - \eta)(\lambda_b - \nu + \eta).
\end{aligned}$$

So for all $i \in \{2, \ldots, n\}$, we find that $\lambda_i = \nu - \eta$. $\square$

Since $\{e_2, \ldots, e_n\}$ are all in the eigenspace of $\nu - \eta$, we may assume that $\{e_2, \ldots, e_n\}$ is in fact the canonical basis of span$\{e_2, \ldots, e_n\}$:

$$C_{2ij} = \lambda_i^{(2)} \delta_{ij},$$

for $2 \leq i, j \leq n$ and for certain eigenvalues $\lambda_i^{(2)}$. Let us set $\nu_1 = \nu$ and $\eta_1 = \eta$. We define

$$\begin{aligned}
\nu_2 &= \frac{1}{2} \lambda_2^{(2)}, \\
\eta_2 &= \frac{1}{2} \sqrt{{\lambda_2^{(2)}}^2 + 8\eta_1(\eta_1 - \nu_1)} = \sqrt{\nu_2^2 + 2\eta_1(\eta_1 - \nu_1)},
\end{aligned}$$

thus ${\eta_2}^2 - {\nu_2}^2 = 2\eta_1(\eta_1 - \nu_1)$. So for $2 \leq a, b, c, d \leq n$ we have

$$\begin{aligned}
0 &= (\eta_1^2 - \nu_1^2)(\delta_{bc}\delta_{ad} - \delta_{ac}\delta_{bd}) + \sum_{i=1}^{n} (C_{bci}C_{adi} - C_{aci}C_{bdi}) \\
&= (\eta_1^2 - \nu_1^2)(\delta_{bc}\delta_{ad} - \delta_{ac}\delta_{bd}) + \sum_{i=2}^{n} (C_{bci}C_{adi} - C_{aci}C_{bdi}) + (\nu_1 - \eta_1)^2 (\delta_{bc}\delta_{ad} - \delta_{ac}\delta_{bd}) \\
&= 2\eta_1(\eta_1 - \nu_1)(\delta_{bc}\delta_{ad} - \delta_{ac}\delta_{bd}) + \sum_{i=2}^{n} (C_{bci}C_{adi} - C_{aci}C_{bdi}) \\
&= (\eta_2^2 - {\nu_2}^2)(\delta_{bc}\delta_{ad} - \delta_{ac}\delta_{bd}) + \sum_{i=2}^{n} (C_{bci}C_{adi} - C_{aci}C_{bdi}).
\end{aligned}$$

This is just (4.3.1) again, but restricted to $\{e_2, \ldots, e_n\}$ and with ν_2 and η_2. So we find that $\lambda_i^{(2)} = \nu_2 - \eta_2$ for $3 \leq i \leq n$. We can continue this process, defining

$$\begin{aligned}
\nu_i &= \frac{1}{2} \lambda_i^{(i)}, \\
\eta_i &= \frac{1}{2} \sqrt{{\lambda_i^{(i)}}^2 + 8\eta_{i-1}(\eta_{i-1} - \nu_{i-1})} = \sqrt{\nu_i^2 + 2\eta_{i-1}(\eta_{i-1} - \nu_{i-1})}.
\end{aligned}$$

We can summarise the results of this process as follows:

Theorem 4.3.2. *If we choose the canonical basis $\{e_1, \ldots, e_n\}$ where*

$$\begin{aligned}
e_i &= \operatorname*{argmax}_{X \in (U_pM)_i} C(X, X, X), \\
\lambda_i^{(i)} &= C(e_i, e_i, e_i),
\end{aligned}$$

where $(U_pM)_i = \{X \in U_pM \mid X \perp e_1, \ldots, f_{e-1}\}$, and we define ν_i and η_i as above. Then C has the following form:

$$\begin{aligned}
C_{iii} &= 2\nu_i & C_{ijj} &= \nu_i - \eta_i \\
C_{iij} &= 0 & C_{ijk} &= 0,
\end{aligned}$$

where $1 \leq i, j, k \leq n$ and $i < j < k$. In particular, C is completely determined by the curvatures c and $\tilde{c}$, and the values $\lambda_i^{(i)}$.

Proof. The process to get this specific form is mentioned above. To see that C depends only on these values, note that we constructed ν_1 and η_1 using c, $\tilde{c}$ and $\lambda_1^{(1)}$. Using the definition, we can write any ν_i and η_i in function of $c, \tilde{c}$ and $\{\lambda_1^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_i^{(i)}\}$. $\square$

A special case occurs when M is a Lagrangian submanifold of constant sectional curvature $\tilde{c}$:

Theorem 4.3.3. *Let M^n be a Lagrangian submanifold of a complex space form $\tilde{M}(4\tilde{c})$. Then M has constant sectional curvature $\tilde{c}$ if and only if, at each point $p \in M$, C satisfies*

$$C_{iii} = \mu_i \qquad C_{ijk} = 0,$$

where i, j, k not all equal, for some functions μ_i and for some orthonormal basis $\{e_1, \ldots, e_n\}$ [Dil+12].

Proof. If M has constant sectional curvature $\tilde{c}$, then take the canonical basis mentioned in theorem 4.3.2. But since $c = \tilde{c}$, we find that $\nu = \eta$ and as a consequence $\nu_i = \eta_i$ for all $1 \leq i \leq n$. Then by theorem 4.3.2, C satisfies the required form by taking $\mu_i = 2\nu_i$.

Conversely, suppose that C satisfies the given form for some basis $\{e_1, \ldots, e_n\}$. The equation of Gauss (4.2.2) reduces to

$$\langle \tilde{c}(e_a \wedge e_b)e_c, e_d\rangle = \langle R(e_a, e_b)e_c, e_d\rangle - \sum_{i=1}^{n} (C_{bci}C_{adi} - C_{aci}C_{bdi}) = 0,$$

so we have to prove this summation vanishes. But both terms in the summation are only nonzero when $a = b = c = d = i$, in which case they equal eachother. So by linearity, we indeed have that $R(X, Y) = \tilde{c}(X \wedge Y)$ for all $X, Y \in T_pM$ and thus M^n has constant sectional curvature $\tilde{c}$. $\square$

Chapter 5

Classification results

In this chapter we give classification results for certain classes of Lagrangian submanifolds. We mention the results without proof or explanation, as these fall beyond the scope of this thesis. References are of course provided for the interested reader.

5.1 Lagrangian surfaces

If $n = 2$, then the conditions of pseudo-parallel cubic form and H-pseudo-parallel are trivial. As can be seen in the summary of constraints on a Lagrangian surface, we end up with 2 "weakest conditions": being H-semi-parallel, having constant curvature $K = c$ or being H-umbilical. Due to theorem 3.2.12, the condition of H-semi-parallel means we locally have one of the other two options.

The Lagrangian surfaces of constant curvature have been classified by Chen:

Theorem 5.1.1. *There exist 19 families of Lagrangian surfaces of constant curvature in $\mathbb{C}^2$. 12 of the 19 families are obtained via Legendre curves. Conversely, Lagrangian surfaces of constant curvature in $\mathbb{C}^2$ can be obtained locally from the 19 families [Che04; Che05d].*

Theorem 5.1.2. *There exist 32 families of Lagrangian surfaces of constant curvature in $\mathbb{C}P^2$. 25 of the 32 families are obtained via Legendre curves. Conversely, Lagrangian surfaces of constant curvature in $\mathbb{C}P^2$ can be obtained locally from the 32 families [Che05c; Che05e; Che06].*

Theorem 5.1.3. *There exist 68 families of Lagrangian surfaces of constant curvature in $\mathbb{C}H^2$. 48 of the 68 families are obtained via Legendre curves. Conversely, Lagrangian surfaces of constant curvature in $\mathbb{C}H^2$ can be obtained locally from the 68 families [Che05b; Che05e; Che07].*

H-umbilical Lagrangian surfaces have been classified by Chen:

Theorem 5.1.4. *Let M be a non-totally geodesic H-umbilical Lagrangian surface in $\mathbb{C}^2$ satisfying*

$$h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_2) = \mu J e_2, \quad h(e_2, e_2) = \mu J e_1,$$

such that the integral curves of e_1 are geodesics in M. Then we have [Che97a; Che99]:

(i) if M is flat, then one of the following 2 cases happens:

(i-1) M is a Lagrangian cylinder, i.e. a cylinder over a curve whose rulings are lines parallel to a fixed plane,

(i-2) M is a twisted product manifold of the form ${}_f I \times \mathbb{E}$,

(ii) if M is of constant nonzero sectional curvature c, then up to rigid motions of $\mathbb{C}^2$, M is a Lagrangian pseudo-sphere, i.e. the complex extensor $F \otimes \iota$ of the unit circle of $\mathbb{E}^2$ via the unit speed curve

$$F(s) = \frac{e^{2\sqrt{c}si} + 1}{2\sqrt{c}i},$$

which is then an immersion of an open portion of the 2-sphere $S^2(c)$,

(iii) if M contains no open subset of constant sectional curvature, then, up to rigid motions of $\mathbb{C}^2$, M is a complex extensor of the unit circle of $\mathbb{E}^2$.

Theorem 5.1.5. *Let M be a non-totally geodesic H-umbilical Lagrangian surface in $\mathbb{C}P^2(4\tilde{c})$ for the immersion $\bar{\psi} : M \to \mathbb{C}P^2$. We have [Che97b]:*

(i) if M has constant sectional curvature c, it belongs to one of the 32 families mentioned before,

(ii) if M contains no open subsets of constant sectional curvature $\geq \tilde{c}$ and if the integral curves of JH are geodesics in M, then there exists a unit speed Legendre curve

$$z(x) = (z_1(x), z_2(x)) : I \to S^3(\tilde{c}) \subset \mathbb{C}^2,$$

such that up to rigid motions of $\mathbb{C}P^2(4\tilde{c})$, $\bar{\psi} : M \to \mathbb{C}P^2(4\tilde{c})$ with $\bar{\psi}$ given by $\Pi \circ \psi$ where ψ is defined by

$$\psi(x, \theta) = (z_1(x), z_2(x) \sin\theta, z_2(x) \cos\theta).$$

Theorem 5.1.6. *Let M be a non-totally geodesic H-umbilical Lagrangian surface in $\mathbb{C}H^2(4\tilde{c})$ for the immersion $\bar{\psi} : M \to \mathbb{C}H^2$. Then we have [Che97b]:*

(i) if M has constant sectional curvature c, it belongs to one of the 68 families mentioned before,

(ii) if M contains no open subsets of constant sectional curvature $\geq \tilde{c}$ and if the integral curves of JH are geodesics in M, then we define

$$k(x) = \frac{\mu'(x)}{\lambda - 2\mu(x)},$$

and

$$u(x) = \tilde{c} + \mu^2(x) + k^2(x).$$

Then one of the following cases happens:

(ii-1) *if* $u(x) > 0$*, there exists a unit speed Legendre curve* $z = (z_1, z_2) : I \to H_1^3(\tilde{c}) \subset \mathbb{C}_1^2$ *such that up to rigid motions of* $\mathbb{C}H^2(4\tilde{c})$*,* $\bar{\psi}$ *is locally given by* $\Pi \circ \psi$ *where*

$$\psi(x, \theta) = (z_1(x), z_2(x) \cos\theta, z_2(x) \sin\theta),$$

(ii-2) *if* $u(x) < 0$*, there exists a unit speed Legendre curve* $z = (z_1, z_2) : I \to H_1^3(\tilde{c}) \subset \mathbb{C}_1^2$ *such that up to rigid motions of* $\mathbb{C}H^2(4\tilde{c})$*,* $\bar{\psi}$ *is locally given by* $\Pi \circ \psi$ *where*

$$\psi(x, \theta) = (z_1(x) \cosh\theta, z_1(x) \sinh\theta, z_2(x)),$$

(ii-3) *if* $u(x) = 0$*, then up to rigid motions of* $\mathbb{C}H^2(4\tilde{c})$*,* $\bar{\psi}$ *is locally given by* $\Pi \circ \psi$ *where*

$$\begin{aligned}\psi(x, \theta) = e^{\int_0^x (i\mu+k)dx} \Bigg(&\frac{1}{\sqrt{-\tilde{c}}} \left(1 - \frac{\tilde{c}\theta^2}{2} - \int_0^x (i\mu + k) e^{-\int_0^t 2k(t)dt} dx \right), \\ &(i\mu(0) - k(0)) \left(\frac{\theta^2}{2} + \frac{1}{\tilde{c}} \int_0^x (i\mu + k) e^{-\int_0^t 2k(t)dt} dx \right), \theta \Bigg).\end{aligned}$$

5.2 H-umbilical Lagrangian submanifolds

Chen did not just classify the H-umbilical Lagrangian surfaces, but in fact the classified the H-umbilical Lagrangian submanifolds as a whole:

Theorem 5.2.1. *Let* M^n $(n \geq 3)$ *be a non-totally geodesic* H*-umbilical Lagrangian submanifold of* $\mathbb{C}^n$*. Then [Che97a; Che99]:*

(i) *if* M *is flat, then one of the following 2 cases happens:*

(i-1) M *is a Lagrangian cylinder, i.e. a cylinder over a curve whose rulings are* $(n-1)$*-planes parallel to a fixed* $(n-1)$*-plane,*

(i-2) M *is a twisted product manifold of the form* ${}_fI \times \mathbb{E}^{n-1}$*,*

(ii) *if* M *is a manifold of constant nonzero sectional curvature* c*, then up to rigid motions of* $\mathbb{C}^n$*,* M *is a Lagrangian pseudo-sphere, i.e. the complex extensor* $F \otimes \iota$ *of the unit hypersphere of* $\mathbb{E}^n$ *via the unit speed curve*

$$F(s) = \frac{e^{2\sqrt{c}si} + 1}{2\sqrt{c}i},$$

which is then an immersion of an open portion of the n*-sphere* $S^n(c)$*,*

(iii) *if* M *contains no open subset of constant sectional curvature, then up to rigid motions of* $\mathbb{C}^n$*,* M *is a complex extensor of the unit hypersphere of* $\mathbb{E}^n$*.*

Theorem 5.2.2. *Let* M $(n \geq 3)$ *be a non-totally geodesic* H*-umbilical Lagrangian submanifold of* $\mathbb{C}P^n(4\tilde{c})$ *for the immersion* $\bar{\psi} : M \to \mathbb{C}P^n$*. Then [Che97b]:*

(i) *if* M *is a manifold of constant sectional curvature* c*, then either:*

(i-1) $c = \tilde{c}$,

(i-2) $c > \tilde{c}$ *and up to rigid motions of* $\mathbb{C}P^n(4\tilde{c})$ *the immersion* $\bar{\psi} = \Pi \circ \psi$ *where*

$$\begin{aligned}\psi(x, y_1, \ldots, y_n) &= \frac{e^{i\sqrt{c-\tilde{c}}x}}{2\sqrt{c}} \Bigg(\Bigg(\frac{\sqrt{c-\tilde{c}}(\sqrt{c-\tilde{c}} - \sqrt{c})}{\sqrt{\tilde{c}}} + \sqrt{\tilde{c}}y_1 \Bigg) e^{i\sqrt{c}x} \\ &+ \Bigg(\frac{\sqrt{c-\tilde{c}}(\sqrt{c-\tilde{c}} + \sqrt{c})}{\sqrt{\tilde{c}}} + \sqrt{\tilde{c}}y_1 \Bigg) e^{-i\sqrt{c}x}, \\ &(\sqrt{c} - \sqrt{c-\tilde{c}} + \sqrt{c-\tilde{c}}y_1)e^{i\sqrt{c}x} - (\sqrt{c} + \sqrt{c-\tilde{c}} - \sqrt{c-\tilde{c}}y_1)e^{-i\sqrt{c}x}, \\ &\sqrt{c}y_2(e^{i\sqrt{c}x} + e^{-i\sqrt{c}x}), \ldots, \sqrt{c}y_n(e^{i\sqrt{c}x} + e^{-i\sqrt{c}x}) \Bigg),\end{aligned}$$

with $y_1^2 + \cdots + y_n^2 = 1$,

(ii) if M *contains no open subsets of constant sectional curvature* $\geq \tilde{c}$, *then there exists a unit speed Legendre curve*

$$z(x) = (z_1(x), z_2(x)) : I \to S^3(\tilde{c}) \subset \mathbb{C}^2,$$

such that up to rigid motions of $\mathbb{C}P^n(4\tilde{c})$, $\bar{\psi} = \Pi \circ \psi$ *where* ψ *is defined by*

$$\psi(x, y_1, \ldots, y_n) = (z_1(x), z_2(x)y_1, \ldots, z_2(x)y_n),$$

with $y_1^2 + \cdots + y_n^2 = 1$.

Theorem 5.2.3. *Let* M $(n \geq 3)$ *be a non-totally geodesic* H*-umbilical Lagrangian submanifold of* $\mathbb{C}H^n(4\tilde{c})$ *for the immersion* $\bar{\psi} : M \to \mathbb{C}H^n$. *Then [Che97b]:*

(i) if M *is a manifold of constant sectional curvature* c, *then either:*

(i-1) $c = \tilde{c}$,

(i-2) $c > \tilde{c}$ *and up to rigid motions of* $\mathbb{C}H^n(4\tilde{c})$ *the immersion* $\bar{\psi} = \Pi \circ \psi$ *where, if* $c > 0$,

$$\begin{aligned}\psi(x, y_1, \ldots, y_n) &= \frac{e^{i\sqrt{c-\tilde{c}}x}}{2\sqrt{c}} \Bigg(\Bigg(\frac{\sqrt{c-\tilde{c}}(\sqrt{c-\tilde{c}} - \sqrt{c})}{\sqrt{-\tilde{c}}} - \sqrt{-\tilde{c}}y_1 \Bigg) e^{i\sqrt{c}x} \\ &+ \Bigg(\frac{\sqrt{c-\tilde{c}}(\sqrt{c-\tilde{c}} + \sqrt{c})}{\sqrt{-\tilde{c}}} - \sqrt{-\tilde{c}}y_1 \Bigg) e^{-i\sqrt{c}x}, \\ &(\sqrt{c} - \sqrt{c-\tilde{c}} + \sqrt{c-\tilde{c}}y_1)e^{i\sqrt{c}x} - (\sqrt{c} + \sqrt{c-\tilde{c}} - \sqrt{c-\tilde{c}}y_1)e^{-i\sqrt{c}x}, \\ &\sqrt{c}y_2(e^{i\sqrt{c}x} + e^{-i\sqrt{c}x}), \ldots, \sqrt{c}y_n(e^{i\sqrt{c}x} + e^{-i\sqrt{c}x}) \Bigg),\end{aligned}$$

with $y_1^2 + \cdots + y_n^2 = 1$; *when* $c = 0$,

$$\psi(x, u_2, \ldots, u_n) = \frac{e^{i\sqrt{-\tilde{c}}x}}{2\sqrt{c}} \left(\frac{1}{\sqrt{\tilde{c}}} - ix + \frac{\sqrt{-\tilde{c}}}{2} \sum_{j=2}^{n} u_j^2, x + \frac{i}{2} \sum_{j=2}^{n} u_j^2, u_2, \ldots, u_n \right),$$

or when $c < 0$,

$$\psi(x,u_2,\ldots,u_n) = \frac{e^{i\sqrt{c-\tilde{c}}x}}{2}\left(\frac{1}{\sqrt{-\tilde{c}}}\left(e^{\sqrt{-c}x}\left(1-\frac{\sqrt{c-\tilde{c}}}{\sqrt{-c}}i-\tilde{c}\sum_{j=2}^{n}u_j^2\right)\right.\right.$$
$$\left.\left.+e^{-\sqrt{-c}x}\left(1+\frac{\sqrt{c-\tilde{c}}}{\sqrt{-c}}i\right)\right), e^{\sqrt{-c}x}\left(\frac{1}{\sqrt{-c}}+(\sqrt{c-\tilde{c}}i-\sqrt{-c})\sum_{j=2}^{n}u_j^2\right)\right.$$
$$\left.-\frac{1}{\sqrt{-c}}e^{-\sqrt{-c}x}, 2u_2e^{\sqrt{-c}x},\ldots,2u_ne^{\sqrt{-c}x}\right),$$

(ii) if M *contains no open subsets of constant sectional curvature* $\geq \tilde{c}$*, then we define*

$$k(x) = \frac{\mu'(x)}{\lambda - 2\mu(x)},$$

and

$$u(x) = \tilde{c} + \mu^2(x) + k^2(x).$$

Then M *is foliated by real space forms* $N^{n-1}(u(x))$ *of constant sectional curvature* $u(x)$*. Then the following cases happen:*

(ii-1) if $u(x) > 0$*, then there exists a unit speed Legendre curve,*

$$z(x) = (z_1(x), z_2(x)) : I \to H_1^3(\tilde{c}) \subset \mathbb{C}_1^2,$$

such that up to rigid motions of $\mathbb{C}H^n(4\tilde{c})$*,* $\bar{\psi} = \Pi \circ \psi$ *where*

$$\psi(x, y_1, \ldots, y_n) = (z_1(x), z_2(x)y_1, \ldots, z_2(x)y_n),$$

with $y_1^2 + \cdots + y_n^2 = 1$*,*

(ii-2) if $u(x) < 0$*, then there exists a unit speed Legendre curve*

$$z(x) = (z_1(x), z_2(x)) : I \to H_1^3(\tilde{c}) \subset \mathbb{C}_1^2,$$

such that up to rigid motions of $\mathbb{C}H^n(4\tilde{c})$*,* $\bar{\psi} = \Pi \circ \psi$ *where*

$$\psi(x, y_1, \ldots, y_n) = (z_1(x)y_1, \ldots, z_1(x)y_n, z_2(x)),$$

with $y_1^2 - y_2^2 - \cdots - y_n^2 = 1$*,*

(ii-3) if $u(x) = 0$*, then up to rigid motions of* $\mathbb{C}H^n(4\tilde{c})$*,* $\bar{\psi} = \Pi \circ \psi$ *is locally given by*

$$\psi(x,u_2,\ldots,u_n) = e^{\int_0^x (i\mu+k)dx}\left(\frac{1}{\sqrt{-\tilde{c}}}\left(1-\frac{\tilde{c}}{2}\sum_{j=2}^{n}u_j^2-\int_0^x (i\mu+k)e^{-\int_0^t 2k(t)dt}dx\right),\right.$$
$$\left.(i\mu(0)-k(0))\left(\frac{1}{2}\sum_{j=2}^{n}u_j^2+\frac{1}{\tilde{c}}\int_0^x (i\mu+k)e^{-\int_0^t 2k(t)dt}dx\right), u_2,\ldots,u_n\right).$$

5.3 Parallel Lagrangian submanifolds in $\mathbb{C}P^n$

Naitoh studied and classified the parallel Lagrangian submanifolds $\mathbb{C}P^n$ in a series of papers [Nai80; Nai81a; Nai81b; Nai83a; Nai83b; NT82]. However, we refer to the classification in [Dil+12] as it is more "geometric":

Theorem 5.3.1. *Let M be a parallel Lagrangian submanifold of $\mathbb{C}P^n$. Then M is one of the following:*

(i) *M is totally geodesic,*

(ii) *M is locally the Calabi product of a point with a lower-dimension parallel Lagrangian submanifold,*

(iii) *M is locally the Calabi product of two lower-dimensional parallel Lagrangian submanifolds,*

(iv) $n = k(k+1)/2 - 1$ *for* $k \geq 3$ *and* M *is congruent with* $SU(k)/SO(k)$,

(v) $n = k^2 - 1$ *for* $k \geq 3$ *and* M *is congruent with* $SU(k)$,

(vi) $n = 2k^2 - k - 1$ *for* $k \geq 3$ *and* M *is congruent with* $SU(2k)/Sp(k)$,

(vii) $n = 26$ *and* M *is congruent with* E_6/F_4.

Part II

δ-invariants of Lagrangian submanifolds

Chapter 6

δ-invariants of Riemannian manifolds

In this chapter, we will discuss Chen's δ-invariants for general Riemannian manifolds and submanifolds, not just for Lagrangian submanifolds of complex space forms. Most of the content from this chapter is adapted from [Che11] and [Che13].

6.1 Introduction

Curvature invariants are the number one Riemannian invariants and the most natural ones. They also play key roles in physics: for instance, the magnitude of a force required to move an object at constant speed, according to Newton's law, a constant multiple of the curvature of the trajectory. The motion of a body in a gravitational field is determined, according to Einstein, by the curvatures of spacetime. All sorts of shapes, from soap bubbles to red blood cells, seem to be determined by various curvatures. Borrow a term from biology, Riemannian invariants are the DNA of Riemannian manifolds. Classically, among the Riemannian curvature invariants, people have been studying sectional, scalar and Ricci curvatures in great detail.

One of the most fundamental problems in the theory of submanifolds is that of immersibility (or non-immersibility) of a Riemannian manifold in a Euclidean space (or, more generally, in a space form). This problem has been around since Bernhard Riemann and was posed explicitly by Ludwig Schläfli in [Sch71]. Schläfli asserted that any Riemannian manifold M^n can be isometrically embedded in Euclidean space of dimension $\frac{1}{2}n(n+1)$. Apparently, it is appropriate to assume that he had in mind of analytic metrics and local analytic embeddings. This was later called *Schläfli's conjecture.*

Maurice Janet published in [Jan26] a proof of Schläfli's conjecture which states that a real analytic Riemannian manifold M^n can be locally isometrically embedded into any real analytic Riemannian manifold of dimension $\frac{1}{2}n(n+1)$. Élie Cartan revised Janet's paper in [Car27]; yet both Janet's and Cartan's proofs contained obscurities. C. Burstin got rid of them in [Bur31]. This result of Cartan-Janet implies that every Einstein manifold M^n ($n \geq 3$) can be locally isometrically embedded in $\mathbb{E}^{n(n+1)/2}$.

The Cartan-Janet theorem is dimension-wise the best possible, i.e. there exist real analytic Riemannian manifolds M^n which do not possess smooth local isometric embeddings into any Euclidean space of dimension strictly less than $\frac{1}{2}n(n+1)$. Not every Riemannian

n-manifold can be isometrically immersed in $\mathbb{E}^m$ with $m \leq \frac{1}{2}n(n+1)$. For instance, not every Riemannian surface M^2 can be isometrically immersed in $\mathbb{E}^3$. A global isometric embedding theorem was proven by John F. Nash in [Nas56]

Theorem 6.1.1. *Every compact Riemannian manifold M^n can be isometrically embedded in any small portion of a Euclidean space $\mathbb{E}^m$ with $m = \frac{1}{2}n(3n+11)$. Every non-compact Riemannian manifold M^n can be isometrically embedded in any small portion of a Euclidean space $\mathbb{E}^m$ with $m = \frac{1}{2}n(n+1)(3n+11)$.*

Robert E. Greene improved Nash's result in [Gre70] and proved that every non-compact Riemannian manifold M^n can be isometrically embedded in the Euclidean space $\mathbb{E}^m$ with $m = 2(2n+1)(3n+7)$. Also, it was proven independently in [Gre70] and [GR70] that a local isometric embedding from a Riemannian manifold M^n into $\mathbb{E}^{\frac{1}{2}n(n+1)+n}$ always exists.

The Nash embedding theorem was aimed at the hope that if Riemannian manifolds could be regarded as Riemannian submanifolds, this would then yield the opportunity to use extrinsic help in the study of (intrinsic) Riemannian geometry. However, this hope had not been materialized according to Mikhail Gromov in [Gro85]. There were several reasons why:

(i) It requires a very large codimension for a Riemannian manifold to admit an isometric embedding in Euclidean spaces in general. But submanifolds of higher codimension are very difficult to be understood, e.g. there are no general results for arbitrary Riemannian submanifolds except the three fundamental equations of Gauss, Codazzi and Ricci.

(ii) As explained in [Yau92], "What is lacking in the Nash theorem is the control of the extrinsic quantities in relation to the intrinsic quantities". In other words, we do not have any optimal relationships between intrinsic and extrinsic invariants.

Since there are no obstructions to isometric embeddings according to Nash's theorem, in order to study isometric immersions (or embeddings), it is natural to impose some suitable constraints. Shiing-Shen Chern asked in [Che68]: "What are necessary conditions for a Riemannian manifold to admit a minimal isometric immersion into a Euclidean space?". From the equation of Gauss, it follows that a necessary condition is that $\mathrm{Ric} \leq 0$ (and in particular, $\tau \leq 0$). For many years, this was the *only* known Riemannian obstruction for a general Riemannian manifold to admit a minimal immersion into a Euclidean space with arbitrary codimension, until Bang-Yen Chen introduced his δ-invariants in [Che93; Che94; Che95]. These invariants were later applied to Lagrangian submanifolds [Che00b].

6.2 Formal definition

Let M^n be a Riemannian manifold. Recall the definition of the scalar curvature:

$$\tau = \sum_{i<j} K(e_i, e_j),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for T_pM.

Definition 6.2.1. Let L be a subspace of T_pM of dimension $2 \leq m \leq n-1$ and $\{e_1, \ldots, e_m\}$ an orthonormal basis of L. We define the *scalar curvature* of L as

$$\tau(L) = \sum_{i<j} K(e_i, e_j).$$

In particular, if $m = 2$, then $\tau(L)$ is the sectional curvature of L.

We will need the following definition:

Definition 6.2.2. We denote by $\mathcal{S}(n)$ the set of k-tuples $(n_1, \ldots, n_k)$ where

(i) $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$,

(ii) $2 \leq n_1 \leq n_2 \leq \ldots \leq n_k < n$,

(iii) $n_1 + \cdots + n_k \leq n$.

Remark 6.2.3. Note that $\#\mathcal{S}(n)$ increases quite rapidly with n: it is equal to $p(n) - 1$ where $p(n)$ is the partition function. The asymptotic behaviour of $\#\mathcal{S}(n)$ is given by

$$\#\mathcal{S}(n) \approx \frac{1}{4n\sqrt{3}} \exp\left(\sqrt{\frac{2n}{3}}\pi\right) \quad \text{as } n \to \infty.$$

Definition 6.2.4. Let $(n_1, \ldots, n_k) \in \mathcal{S}(n)$. We define the *$\delta$-invariant* of $(n_1, \ldots, n_k)$ at a point $p \in M$ as

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \ldots + \tau(L_k)\},$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM with $\dim_{\mathbb{R}} L_j = n_j$.

Because of a compactness argument, this infimum will always be reached and is actually a minimum. The subspaces $\{L_1, \ldots, L_k\}$ for which this minimum is attained need not be unique, however.

Proposition 6.2.5. *For certain integers, the delta-invariant has a well-known meaning:*

(i) $\delta(\emptyset) = \tau$,

(ii) $\delta(2) = \tau - \inf_{\pi \in T_pM} \tilde{K}(\pi)$,

(iii) $\delta(n-1) = \max_{\|X\|=1} \operatorname{Ric}(X)$.

Proof. Items (i) and (ii) follow immediately from the definition of the δ-invariants. For item (iii), let $L = \operatorname{span}\{e_1, \ldots, e_{n-1}\}$ be the subspace of T_pM minimizing $\tau(L)$. Now,

$$\tau - \tau(L) = \sum_{i<j}^{n} K(e_i, e_j) - \sum_{i<j}^{n-1} K(e_i, e_j) = \sum_{i=1}^{n-1} K(e_i, e_n) = \operatorname{Ric}(e_n).$$

Since L is the subspace minimizing the left-hand side of the above equality, e_n must be the unit vector maximizing the right-hand side. $\square$

6.3 Optimal general inequality for δ-invariants

Before stating any results, we will introduce some notation. For a given delta-invariant $\delta(n_1, \ldots, n_k)$ on a Riemannian manifold M^n and a point $p \in M$, we consider the mutually orthogonal subspaces $L_1, \ldots, L_k$ with $\dim(L_i) = n_i$ of T_pM, minimizing the infimum $\inf\{\tau(L_1) + \ldots + \tau(L_k)\}$. We then choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM such that

$$\begin{aligned}
&e_1, \ldots, e_{n_1} \in L_1,\\
&e_{n_1+1}, \ldots, e_{n_1+n_2} \in L_2,\\
&\;\vdots\\
&e_{n_1+\cdots+n_{k-1}+1}, \ldots, e_{n_1+\cdots+n_k} \in L_k,
\end{aligned}$$

and we shall define L_{k+1} as the subspace of dimension $n_{k+1} = n - n_1 - \cdots - n_k$, spanned by $\{e_{n_1+\cdots+n_k+1}, \ldots, e_n\}$. Then we have that $T_pM = L_1 \oplus L_2 \oplus \cdots \oplus L_{k+1}$. Note that L_{k+1} may be empty and thus n_{k+1} may be zero.

To make notation with these indices a bit more easy, we set

$$\begin{aligned}
&\Delta_1 = \{1, \ldots, n_1\},\\
&\Delta_2 = \{n_1 + 1, \ldots, n_1 + n_2\},\\
&\;\vdots\\
&\Delta_k = \{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_k\},\\
&\Delta_{k+1} = \{n_1 + \cdots + n_k + 1, \ldots, n\},
\end{aligned}$$

so that we can apply following conventions for the ranges of summation indices:

$$a, b, c \in \{1, \ldots, n\}, \quad i, j \in \{1, \ldots, k\}, \quad \alpha_i, \beta_i, \gamma_i \in \Delta_i, \quad r, s, t \in \Delta_{k+1}.$$

Definition 6.3.1. For each $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we define the functions $c(n_1, \ldots, n_k)$ and $b(n_1, \ldots, n_k)$ by

$$\begin{aligned}
c(n_1, \ldots, n_k) &= \frac{n^2(n + k - 1 - \sum_{j=1}^k n_j)}{2(n + k - \sum_{j=1}^k n_j)},\\
b(n_1, \ldots, n_k) &= \frac{1}{2}n(n-1) - \frac{1}{2}\sum_{j=1}^k n_j(n_j - 1).
\end{aligned}$$

We have the following optimal general inequality [Che05a]:

Theorem 6.3.2. *Let $\phi : M^n \to \tilde{M}^m$ be an isometric immersion of Riemannian manifolds. Then for each k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ we have*

$$\delta(n_1, \ldots, n_k) \leq c(n_1, \ldots, n_k)\|H\|^2 + b(n_1, \ldots, n_k) \max \tilde{K}, \tag{6.3.1}$$

where $\max K$ *is the maximum of the sectional curvature of $\tilde{M}$ restricted to* 2*-plane sections of the tangent space T_pM.*

The equality case holds at $p \in M$ if and only if for any k mutual orthogonal subspaces $\{L_1, \ldots, L_k\}$ of T_pM satisfying the infimum in the definition of $\delta(n_1, \ldots, n_k)$, we have:

(i) The shape operator A at p, with respect to the orthonormal basis $\{e_1, \dots, e_n\}$, takes the form

$$A_\xi = \begin{pmatrix} A_\xi^{(1)} & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A_\xi^{(k)} & \\ & 0 & & \mu_\xi \operatorname{Id}_{n_{k+1}} \end{pmatrix}, \tag{6.3.2}$$

for any normal vector ξ where $A_\xi^{(j)}$ is a symmetric $n_j \times n_j$ submatrix and $\mu_\xi \in \mathbb{R}$ such that

$$\operatorname{trace}(A_\xi^{(1)}) = \cdots = \operatorname{trace}(A_\xi^{(k)}) = \mu_\xi. \tag{6.3.3}$$

(ii) With respect to the orthonormal basis $\{e_1, \dots, e_n\}$, we have that $\tilde{K}(e_{\alpha_i}, e_{\alpha_j}) = \tilde{K}(e_r, e_s) = \max \tilde{K}(p)$ for $i \neq j$, $r \neq s$.

An important special case of this theorem is the following:

Theorem 6.3.3. *Given an n-dimensional submanifold M in a real space form $\tilde{M}^m(\tilde{c})$. Then for each k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$ we have*

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)\|H\|^2 + b(n_1, \dots, n_k)\tilde{c}. \tag{6.3.4}$$

The equality case of (6.3.4) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1, \dots, e_n$ of T_pM such that the shape operator at p satisfies* (6.3.2) *and* (6.3.3)*.*

Because the proof of theorem 6.3.2 is based on the equation of Gauss and since this equation for a totally real submanifold in a complex space form is the same as for a submanifold in a real space form, we obtain the following result:

Theorem 6.3.4. *Let M be an n-dimensional totally real submanifold of a complex space form $\tilde{M}^m(4\tilde{c})$. Then for any k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$ we have*

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)\|H\|^2 + b(n_1, \dots, n_k)\tilde{c}. \tag{6.3.5}$$

The equality case of (6.3.5) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1, \dots, e_n$ of T_pM such that the shape operator at p satisfies* (6.3.2) *and* (6.3.3)*.*

So in particular, this theorem holds for Lagrangian submanifolds of complex space forms.

Definition 6.3.5. Let $\phi : M^n \to \tilde{M}^m$ be an isometric immersion of Riemannian manifolds. If this immersion satisfies the equality in (6.3.1) for some $(n_1, \dots, n_k)$ at every point $p \in M$, then it is called an *ideal immersion*.

The inequality is called "optimal" because ideal immersions that are non-minimal, exist.

Example 6.3.6. Consider the spherical cylinder $M^n = \mathbb{E}^k \times S^{n-k}(1)$ where $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Then at a point $p = (x, y)$, we can consider the tangent space $T_pM = T_x\mathbb{E}^k \oplus T_yS^{n-k}$. So let $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_pM where $\{e_1, \dots, e_k\}$ are the lifts of an orthonormal basis of $T_x\mathbb{E}^k$ and $\{e_{k+1}, \dots, e_n\}$ are the lifts of an orthonormal basis of T_yS^{n-k}. Then we have that

$$K(e_i, e_j) = \begin{cases} 1 & \text{if } k+1 \leq i, j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

So we find that the scalar curvature of M is

$$\tau = \frac{(n-k)(n-k-1)}{2}.$$

Now let $(n_1, \dots, n_l)$ be such that $(n_1 - 1) + \cdots - (n_l - 1) = k$. Then for orthogonal spaces $\{L_1, \dots, L_l\}$, choose each L_i to be the span of n_i vectors in $\{e_1, \dots, e_k\}$ and 1 vector in $\{e_{k+1}, \dots, e_n\}$. Then clearly $\tau(L_i) = 0$ for all these spaces, and thus

$$\delta(n_1, \dots, n_l) = \tau = \frac{(n-k)(n-k-1)}{2}.$$

Now immerse this spherical cylinder as a hypersurface with the immersion

$$\phi : M^n = \mathbb{E}^k \times S^{n-k}(1) \to \mathbb{E}^{n+1} = \mathbb{E}^k \times \mathbb{E}^{k-1}.$$

It is well-known this immersion has $\|H\| = \frac{n-k}{n}$. We can calculate $c(n_1, \dots, n_l)$ to be

$$c(n_1, \dots, n_l) = \frac{n^2(n-k-1)}{2(n-k)},$$

and therefore, for any such $(n_1, \dots, n_l)$, we satisfy the equality in (6.3.1). So ϕ is an ideal immersion.

6.4 Corollaries

Recall from proposition 6.2.5 that $\delta(\emptyset)$ and $\delta(n-1)$ have specific interpretation in terms of the scalar curvature τ and the Ricci curvature Ric. It is natural to wonder what the optimal general inequality tells us about them.

Corollary 6.4.1. *Let M^n be a submanifold of a real space form $\tilde{M}^m(\tilde{c})$ or a totally real submanifold of a complex space form $\tilde{M}^m(4\tilde{c})$. Then*

$$\tau \leq \frac{n(n-1)}{2}(\|H\|^2 + \tilde{c}),$$

with equality holding at a point $p \in M$ if and only if p is a totally umbilical point.

Proof. The inequality follows directly from calculating $c(\emptyset)$ and $b(\emptyset)$. In the equality case, (6.3.2) becomes $A_\xi = \mu_\xi \,\mathrm{Id}$, which is the condition for being totally umbilical. $\square$

Corollary 6.4.2. *Let M^n be a submanifold of a real space form $\tilde{M}^m(\tilde{c})$ or a totally real submanifold of a complex space form $\tilde{M}^m(4\tilde{c})$. Then*

$$\mathrm{Ric}(X) \leq \frac{n^2}{4}\|H\|^2 + (n-1)\tilde{c}.$$

Proof. From calculating $c(n-1)$ and $b(n-1)$, we find that

$$\mathrm{Ric}(X) \leq \max_{\|Y\|=1} \mathrm{Ric}(Y) = \delta(n-1) \leq \frac{n^2}{4}\|H\|^2 + (n-1)\tilde{c},$$

which proves the corollary. □

Finally, in the case of $\delta(2)$, we have an interesting result for the form of the shape operator in the equality case:

Corollary 6.4.3. *Let M^n be a submanifold of a real space form $\tilde{M}^m(\tilde{c})$ or a totally real submanifold of a complex space form $\tilde{M}^m(4\tilde{c})$. Then*

$$\delta(2) \leq \frac{n-2}{2}\left(\frac{n^2}{n-1}\|H\|^2 + (n+1)\tilde{c}\right),$$

and equality holds at a point $p \in M$ if and only if there is a basis $\{e_1, \ldots, e_n\}$ of T_pM such that the shape operator A_ξ has the form

$$A_\xi = \mu_\xi \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{n-1} \end{pmatrix}, \tag{6.4.1}$$

where $\mu_\xi \in \mathbb{R}$.

Proof. The inequality follows directly from calculating $c(2)$ and $b(2)$. For the equality case, we refer to [Opr08]. □

Chapter 7

Improved inequality for Lagrangian submanifolds

In this chapter, we will show that the general inequality is no longer optimal when restricted to Lagrangian submanifolds. An improved and once again optimal inequality will be given and proven in detail.

7.1 Loss of optimality

Naturally, we can apply theorem 6.3.4 to the case of Lagrangian submanifolds. However, while inequality (6.3.5) is optimal in general, is it still optimal when restricted to the class of Lagrangian submanifolds? It turns out it is not:

Theorem 7.1.1. *Every Lagrangian submanifold satisfying inequality* (6.3.5) *is a minimal submanifold [Che00a].*

Proof. Let M^n be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4\tilde{c})$ satisfying equality in (6.3.5). Then there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM at each $p \in M$ such that, for each normal vector ξ at p, the shape operator A_ξ with respect to $\{e_1, \ldots, e_n\}$ takes the form (6.3.2) and that equation (6.3.3) holds.

If $h \equiv 0$ there is nothing to prove. So assume $h \not\equiv 0$. From (6.3.2), we find that for any $X \in T_pM$ and for any $i \in \{1, \ldots, k\}$, $A_{JX}L_i \subset L_i$. Thus $A_{JL_j}L_i \subset L_i$, and likewise $A_{JL_i}L_j \subset L_j$. From the symmetry of the shape operator we know that $A_{JL_i}L_j = A_{JL_j}L_i$; thus $A_{JL_i}L_j \subset (L_i \cap L_j)$, but since L_i and L_j are orthogonal if $i \neq j$, we find

$$h(L_i, L_i) \subset JL_i, \quad h(L_i, L_j) = 0, \quad i \neq j.$$

Let us remark that

$$\begin{aligned}\langle \operatorname*{trace}_{L_i} h, JX\rangle &= \sum_{\alpha_i \in \Delta_i} \langle h(e_{\alpha_i}, e_{\alpha_i}), JX\rangle = \sum_{\alpha_i \in \Delta_i} \langle A_{Je_{\alpha_i}} e_{\alpha_i}, X\rangle = \sum_{\alpha_i \in \Delta_i} \langle A_{JX} e_{\alpha_i}, e_{\alpha_i}\rangle \\ &= \operatorname{trace} A^{(i)}_{JX},\end{aligned}$$

and

$$\langle \operatorname*{trace}_{L_{k+1}} h, JX\rangle = \sum_{r\in\Delta_{k+1}} \langle h(e_r,e_r), JX\rangle = \sum_{r\in\Delta_{k+1}} \langle A_{Je_r}e_r, X\rangle = \sum_{r\in\Delta_{k+1}} \langle A_{JX}e_r, e_r\rangle$$
$$= \sum_{r\in\Delta_{k+1}} \mu_{JX}\langle e_r, e_r\rangle = n_{k+1}\mu_{JX}.$$

We consider two cases:

(i) $n_1+\ldots+n_k = n$. Then $n_{k+1}=0$ and by the definition of $\mathcal{S}(n)$, we know that $k\geq 2$. We show that $\operatorname{trace} A^{(i)}_{Je_a} = 0$, for any $i\in\{1,\ldots,k\}$. There are two possible situations:

(i-1) $a=\alpha_j$ with $i\neq j$. Then

$$\operatorname{trace} A^{(i)}_{Je_{\alpha_j}} = \sum_{\alpha_i\in\Delta_i} \langle A_{Je_{\alpha_j}}e_{\alpha_i}, e_{\alpha_i}\rangle = \sum_{\alpha_i\in\Delta_i} \langle A_{Je_{\alpha_i}}e_{\alpha_i}, e_{\alpha_j}\rangle = 0,$$

since $A_{Je_{\alpha_i}}e_{\alpha_i}\in L_i$ and $e_{\alpha_j}\in L_j$,

(i-2) $a=\alpha_i\in\Delta_i$. Since $k\geq 2$, there exists a $L_j\perp L_i$. Then by (6.3.3) and (i-1), we find

$$\operatorname{trace} A^{(i)}_{Je_{\alpha_i}} = \operatorname{trace} A^{(j)}_{Je_{\alpha_i}} = 0.$$

Thus combining (i-1) and (i-2), we find that $\operatorname{trace}_{L_i} h = 0$ for all $i\in\{1,\ldots,k\}$.

(ii) $n_1+\cdots+n_k<n$. Then Δ_{k+1} is nonempty, so choose some $r\in\Delta_{k+1}$. Then by (6.3.2) we have that for any a and any $b\neq r$,

$$\langle A_{Je_r}e_a, e_b\rangle = \langle A_{Je_a}e_r, e_b\rangle = \mu_{Je_a}\langle e_r, e_b\rangle = 0. \tag{7.1.1}$$

Now consider two situations:

(ii-1) take $a=r$ in (7.1.1), then we get for any $b\neq r$:

$$0 = \langle A_{Je_r}e_r, e_b\rangle = \langle A_{Je_b}e_r, e_r\rangle = \mu_{Je_b}\langle e_r, e_r\rangle = \mu_{Je_b},$$

(ii-2) by definition, we know that Δ_1 is nonempty. Now consider $a=b=\alpha_1\in\Delta_1$ in (7.1.1) and then take the sum over all α_1, then we find by (6.3.3):

$$0 = \sum_{\alpha_1\in\Delta_1} \langle A_{Je_r}e_{\alpha_1}, e_{\alpha_1}\rangle = \operatorname{trace} A^{(1)}_{Je_r} = \mu_{Je_r}.$$

By combining (ii-1) and (ii-2), we find that $\mu_{JX}=0$ for all $X\in T_pM$. Thus

$$\langle \operatorname*{trace}_{L_i} h, JX\rangle = \operatorname{trace} A^{(i)}_{JX} = \mu_{JX} = 0,$$
$$\langle \operatorname*{trace}_{L_{k+1}} h, JX\rangle = n_{k+1}\mu_{JX} = 0.$$

So the trace of h over any space in $\{L_1,\ldots,L_k,L_{k+1}\}$ vanishes.

Now the mean curvature is

$$H = \frac{1}{n}\operatorname{trace} h = \frac{1}{n}\sum_{i=1}^{k+1} \operatorname*{trace}_{L_i} h = 0,$$

so indeed M is minimal. □

7.2 Improved inequality

Since every Lagrangian submanifold satisfying the equality in (6.3.5) is minimal, the inequality is not optimal for Lagrangian manifolds. So we should be able to find a function $c_L(n_1, \ldots, n_k) \leq c(n_1, \ldots, n_k)$, such that the inequality with c_L instead of c is optimal. This was done in [CD11b], however as pointed out in [CD11a] there was a mistake in the proof. This was rectified in [Che+13].

We will first give two lemmas:

Lemma 7.2.1. *For any set of real numbers $\{C'_{abc} \mid 1 \leq a, b, c \leq n\}$, which is symmetric in the three indices a, b and c, there exists a Lagrangian isometric immersion $F : U \subset \mathbb{R}^n \to \mathbb{C}^n$ and a point $p \in U$ such that the cubic form of F at p is given by $C_{abc} = C'_{abc}$ where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$ and J is the standard complex structure of $\mathbb{C}^n$.*

Proof. Let $f : U \subset \mathbb{R}^n \to \mathbb{R} : (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$ be a smooth function on an open subset U of $\mathbb{R}^n$. Then

$$F : U \subset \mathbb{R}^n \to \mathbb{C}^n : (x_1, \ldots, x_n) \mapsto \left(x_1 + i\frac{\partial f}{\partial x_1}, \ldots, x_n + i\frac{\partial f}{\partial x_n} \right),$$

is a Lagrangian isometric immersion satisfying $C_{abc} = \frac{\partial^3 f}{\partial x_a \partial x_b \partial x_c}$ at every point of U. For a given set of real numbers $\{C'_{abc} \mid 1 \leq a, b, c \leq n\}$ we can easily construct a smooth function f which satisfies this. For indices $1 \leq a \leq b \leq c \leq n$, consider the polynomial

$$f(x) = \sum_{a<b<c} C_{abc} x_a x_b x_c + \sum_{a=b<c} C_{aac} \frac{1}{2} x_a^2 x_c + \sum_{a<b=c} C_{abb} \frac{1}{2} x_a x_b^2 + \sum_{a=b=c} C_{aaa} \frac{1}{6} x_a^3.$$

This function indeed has the required 3rd order derivatives. □

Lemma 7.2.2. *For real numbers $A_1, \ldots, A_k$, denote by $\Delta(A_1, \ldots, A_k)$ the determinant of the matrix with $A_1, \ldots, A_k$ on the diagonal and all other entries equal to 1:*

$$\Delta(A_1, \ldots, A_k) = \begin{vmatrix} A_1 & 1 & \cdots & 1 & 1 \\ 1 & A_2 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & A_{k-1} & 1 \\ 1 & 1 & \cdots & 1 & A_k \end{vmatrix}, \tag{7.2.1}$$

Then

$$\Delta(A_1, \ldots, A_k) = \prod_{i=1}^{k} (A_i - 1) + \sum_{i=1}^{k} \prod_{j \neq i} (A_j - 1). \tag{7.2.2}$$

In particular, if none of the numbers $A_1, \ldots, A_k$ equal 1, then

$$\Delta(A_1, \ldots, A_k) = \left(1 + \frac{1}{A_1 - 1} + \cdots + \frac{1}{A_k - 1} \right) (A_1 - 1) \ldots (A_k - 1).$$

Proof. We first verify if the result is true for $k = 1$ and $k = 2$ by direct computation of (7.2.1) and (7.2.2).

For the $k = 1$ case, clearly $\Delta(A_1) = A_1$, and equation (7.2.2) gives $(A_1 - 1) + 1 = A_1$ so this case works out fine.

For the $k = 2$ case, $\Delta(A_1, A_2) = A_1A_2 - 1$. Equation (7.2.2) becomes

$$\begin{aligned}
&(A_1 - 1)(A_2 - 1) + (A_1 - 1) + (A_2 - 1) \\
&= A_1A_2 - A_1 - A_2 + 1 + A_1 - 1 + A_2 - 1 \\
&= A_1A_2 - 1,
\end{aligned}$$

so the lemma is also true for $k = 2$.

Now assume that $k \geq 3$ and the theorem holds for $\Delta(A_1, \dots, A_l)$ with $l < k$. We compute the determinant $\Delta(A_1, \dots, A_k)$ by first replacing the k-th column by the k-th column minus the $(k-1)$-th column, then replacing the k-th row by the k-th row minus the $(k-1)$-th row and finally developing the determinant with respect to the last column.

$$\begin{aligned}
\Delta(A_1, \dots, A_k) &= \begin{vmatrix} A_1 & 1 & \cdots & 1 & 0 \\ 1 & A_2 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & A_{k-1} & 1 - A_{k-1} \\ 1 & 1 & \cdots & 1 & A_k - 1 \end{vmatrix} \\
&= \begin{vmatrix} A_1 & 1 & \cdots & 1 & 0 \\ 1 & A_2 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & A_{k-1} & 1 - A_{k-1} \\ 0 & 0 & \cdots & 1 - A_{k-1} & A_k + A_{k-1} - 2 \end{vmatrix} \\
&= (A_k + A_{k-1} - 2)\Delta(A_1, \dots, A_{k-1}) - (A_{k-1})^2\Delta(A_1, \dots, A_{k-2}).
\end{aligned}$$

It is now sufficient to verify that the expression (7.2.2) indeed satisfies the recursion relation

$$\Delta(A_1, \dots, A_k) = (A_k + A_{k-1} - 2)\Delta(A_1, \dots, A_{k-1}) - (A_{k-1})^2\Delta(A_1, \dots, A_{k-2}).$$

Now,

$$
\begin{aligned}
&(A_k + A_{k-1} - 2)\left(\prod_{i=1}^{k-1}(A_i - 1) + \sum_{i=1}^{k-1}\prod_{j\neq i}^{k-1}(A_j - 1)\right) \\
&\quad - (A_{k-1})^2\left(\prod_{i=1}^{k-2}(A_i - 1) + \sum_{i=1}^{k-2}\prod_{j\neq i}^{k-2}(A_j - 1)\right) \\
&= \prod_{i=1}^{k}(A_i - 1) + (A_{k-1} - 1)\prod_{i=1}^{k-1}(A_i - 1) + \sum_{i=1}^{k-1}\prod_{j\neq i}^{k}(A_j - 1) \\
&\quad + (A_{k-1})\sum_{i=1}^{k-1}\prod_{j\neq i}^{k-1}(A_j - 1) - (A_{k-1} - 1)\prod_{i=1}^{k-1}(A_i - 1) - \sum_{i=1}^{k-2}\prod_{j\neq i}^{k-1}(A_j - 1) \\
&= \prod_{i=1}^{k}(A_i - 1) + \sum_{i=1}^{k}\prod_{j\neq i}^{k}(A_j - 1),
\end{aligned}
$$

which proves the lemma. □

Theorem 7.2.3. *Let M^n be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4\tilde{c})$. Let $(n_1, \ldots, n) \in \mathcal{S}(n)$ be integers satisfying $n_1 + \cdots + n_k < n$ and let $n_{k+1} = n - n_1 - \cdots - n_k$. Then, at any point of M^n, we have*

$$
\delta(n_1, \ldots, n_k) \leq \frac{n^2\left(n_{k+1} + 3k - 1 - 6\sum_{i=1}^{k}\frac{1}{2+n_i}\right)}{2\left(n_{k+1} + 3k + 2 - 6\sum_{i=1}^{k}\frac{1}{2+n_i}\right)}\|H\|^2 + b(n_1, \ldots, n_k)\tilde{c}. \tag{7.2.3}
$$

Assume that equality holds at a point $p \in M^n$. Then with the choice of basis and the notations introduced earlier in this chapter, one has

- *(i) $C_{abc} = 0$ if a, b, c are mutually different and not all in the same Δ_i with $i \in \{1, \ldots, k\}$,*
- *(ii) $C_{\alpha_i\alpha_j\alpha_j} = C_{\alpha_i\alpha_{k+1}\alpha_{k+1}} = \sum_{\beta_i\in\Delta_i} C_{\alpha_i\beta_i\beta_i} = 0$ for $i \neq j$,*
- *(iii) $C_{rrr} = 3C_{rss} = (n_i + 2)C_{\alpha_i\alpha_i r}$ for $r \neq s$.*

Proof. The proof consists of four steps.

Step 1: Set-up. Fix a delta-invariant $\delta(n_1, \ldots, n_k)$ and a point $p \in M^n$. Take linear subspaces $\{L_1, \ldots, L_k\}$ of T_pM and orthonormal basis $\{e_1, \ldots, e_n\}$ described above. The equation of Gauss (1.3.7) gives us

$$
\langle R(e_i, e_j)e_j, e_i\rangle = \tilde{c} + \sum_k (C_{iik}C_{jjk} - C_{ijk}^2).
$$

Then by taking the sum over all indices $b < c$ we get

$$
\tau = \frac{n(n-1)}{2}\tilde{c} + \sum_a \sum_{b<c}(C_{abb}C_{acc} - C_{abc}^2),
$$

and in particular, for each subspace L_i we find

$$\tau(L_i) = \frac{n_i(n_i-1)}{2}\tilde{c} + \sum_a \sum_{\alpha_i<\beta_i} (C_{a\alpha_i\alpha_i}C_{a\beta_i\beta_i} - C^2_{a\alpha_i\beta_i}).$$

Combining these, we have

$$\begin{aligned}\tau - \sum_i \tau(L_i) = &\sum_a \left(\sum_{b<c} (C_{abb}C_{acc} - C^2_{abc}) - \sum_i \sum_{\alpha_i<\beta_i} (C_{a\alpha_i\alpha_i}C_{a\beta_i\beta_i} - C^2_{a\alpha_i\beta_i}) \right) \\ &+ b(n_1,\ldots,n_k)\tilde{c} \\ = &\sum_a \left(\sum_{r<s} \left(C_{arr}C_{ass} - C^2_{ars}\right) + \sum_i \sum_{\alpha_i,r} \left(C_{a\alpha_i\alpha_i}C_{arr} - C^2_{a\alpha_i r}\right) \right. \\ &\left. + \sum_{i<j} \sum_{\alpha_i,\alpha_j} \left(C_{a\alpha_i\alpha_i}C_{a\alpha_j\alpha_j} - C^2_{a\alpha_i\alpha_j}\right) \right) + b(n_1,\ldots,n_k)\tilde{c}.\end{aligned} \tag{7.2.4}$$

Now, let us consider the quadratic terms in the summations:

$$\sum_a \left(\sum_{r<s} C^2_{ars} + \sum_i \sum_{\alpha_i,r} C^2_{a\alpha_i r} + \sum_{i<j} \sum_{\alpha_i,\alpha_j} C^2_{a\alpha_i\alpha_j} \right) \geq \sum_r \sum_{b\neq r} C^2_{brr} + \sum_i \sum_{\alpha_i} \sum_{b\notin\Delta_i} C^2_{b\alpha_i\alpha_i}, \tag{7.2.5}$$

since every term on the right-hand side is also a term on the left-hand side, but not vice-versa. We then find by combining (7.2.4) and (7.2.5) that

$$\begin{aligned}\tau - \sum_i \tau(L_i) \leq &\sum_a \left(\sum_{r<s} C_{arr}C_{ass} + \sum_i \sum_{\alpha_i,r} C_{a\alpha_i\alpha_i}C_{arr} + \sum_{i<j} \sum_{\alpha_i,\alpha_j} C_{a\alpha_i\alpha_i}C_{a\alpha_j\alpha_j} \right) \\ &- \sum_r \sum_{b\neq r} C^2_{brr} - \sum_i \sum_{\alpha_i} \sum_{b\notin\Delta_i} C^2_{b\alpha_i\alpha_i} + b(n_1,\ldots,n_k)\tilde{c}.\end{aligned} \tag{7.2.6}$$

Now, we want to find c_L such that (7.2.6) is less than or equal to

$$n^2 c_L(n_1,\ldots,n_k)\|H\|^2 + b(n_1,\ldots,n_k)\tilde{c} = c_L(n_1,\ldots,n_k) \sum_a \left(\sum_b C_{abb} \right)^2 + b(n_1,\ldots,n_k)\tilde{c}, \tag{7.2.7}$$

such that the value for c_L is the best possible one in the sense that the inequality in the theorem will no longer be true in general for smaller values of c_L. In view of Lemma 1, we have to find the smallest possible c_L for which the following two statements hold:

(I) for any $l \in \{1,\ldots,k\}$ and any $\gamma_l \in \Delta_l$,

$$\begin{aligned}&\sum_{r<s} C_{\gamma_l rr}C_{\gamma_l ss} + \sum_i \sum_{\alpha_i,r} C_{\gamma_l\alpha_i\alpha_i}C_{\gamma_l rr} + \sum_{i<j} \sum_{\alpha_i,\alpha_j} C_{\gamma_l\alpha_i\alpha_i}C_{\gamma_l\alpha_j\alpha_j} \\ &- \sum_r C^2_{\gamma_l rr} - \sum_{i\neq l} \sum_{\alpha_i} C^2_{\gamma_l\alpha_i\alpha_i} \leq c_L(n_1,\ldots,n_k) \left(\sum_b C_{\gamma_l bb} \right)^2,\end{aligned}$$

(II) for any $t \in \Delta_{k+1}$,

$$\sum_{r<s} C_{trr}C_{tss} + \sum_i \sum_{\alpha_i, r} C_{t\alpha_i\alpha_i}C_{trr} + \sum_{i<j} \sum_{\alpha_i,\alpha_j} C_{t\alpha_i\alpha_i}C_{t\alpha_j\alpha_j}$$
$$- \sum_{r\neq t} C_{trr}^2 - \sum_i \sum_{\alpha_i} C_{t\alpha_i\alpha_i}^2 \leq c_L(n_1, \dots, n_k) \left(\sum_b C_{tbb}\right)^2.$$

Step 2: Finding the best possible c_L in (I).

We use that

$$\sum_b C_{\gamma_l bb} = \sum_{i\neq l} \sum_{\alpha_i} C_{\alpha_i\alpha_i\gamma_l}^2 + \sum_{\alpha_l} C_{\alpha_l\alpha_l\gamma_l}^2 + 2\sum_{i<j} \sum_{\alpha_i,\alpha_j} C_{\alpha_i\alpha_i\gamma_l}C_{\alpha_j\alpha_j\gamma_l}$$
$$+ 2\sum_i \sum_{\alpha_i<\beta_i} C_{\alpha_i\alpha_i\gamma_l}C_{\beta_i\beta_i\gamma_l} + 2\sum_i \sum_{\alpha_i,r} C_{\alpha_i\alpha_i\gamma_l}C_{rr\gamma_l} + 2\sum_{r<s} C_{rr\gamma_l}C_{ss\gamma_l} + \sum_r C_{rr\gamma_l}^2.$$

So we can rearrange (I) into

$$(c_L+1)\sum_{i\neq l} \sum_{\alpha_i} C_{\alpha_i\alpha_i\gamma_l}^2 + c_L \sum_{\alpha_l} C_{\alpha_l\alpha_l\gamma_l}^2 + (c_L+1)\sum_r C_{rr\gamma_l}^2 + 2c_L \sum_i \sum_{\alpha_i<\beta_i} C_{\alpha_i\alpha_i\gamma_l}C_{\beta_i\beta_i\gamma_l}$$
$$+ (2c_L-1)\left(\sum_{i<j} \sum_{\alpha_i,\alpha_j} C_{\alpha_i\alpha_i\gamma_l}C_{\alpha_j\alpha_j\gamma_l} + \sum_i \sum_{\alpha_i,r} C_{\alpha_i\alpha_i\gamma_l}C_{rr\gamma_l} + \sum_{r<s} C_{rr\gamma_l}C_{ss\gamma_l}\right) \geq 0.$$

Now, if we put $x_a = C_{aa\gamma_l}$ for all $a = 1, \dots, n$, we can consider the left-hand side of the previous inequality as a quadratic form on $\mathbb{R}^n$. So we need to find necessary and sufficient conditions on c_L for this quadratic form to be non-negative. Two times the matrix of this quadratic form consists of $(k+1)^2$ blocks:

$$M_l = (\Lambda_{ij})_{i,j=1,\dots,k+1},$$

with

$$\Lambda_{ll} = \begin{pmatrix} 2c_L & \cdots & 2c_L \\ \vdots & \ddots & \vdots \\ 2c_L & \cdots & 2c_L \end{pmatrix} \in \mathbb{R}^{n_l\times n_l},$$

$$\Lambda_{k+1\,k+1} = \begin{pmatrix} 2(c_L+1) & 2c_L-1 & \cdots & 2c_L-1 & 2c_L-1 \\ 2c_L-1 & 2(c_L+1) & \cdots & 2c_L-1 & 2c_L-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_L-1 & 2c_L-1 & \cdots & 2(c_L+1) & 2c_L-1 \\ 2c_L-1 & 2c_L-1 & \cdots & 2c_L-1 & 2(c_L+1) \end{pmatrix} \in \mathbb{R}^{n_{k+1}\times n_{k+1}},$$

$$\Lambda_{ii} = \begin{pmatrix} 2(c_L+1) & 2c_L & \cdots & 2c_L & 2c_L \\ 2c_L & 2(c_L+1) & \cdots & 2c_L & 2c_L \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_L & 2c_L & \cdots & 2(c_L+1) & 2c_L \\ 2c_L & 2c_L & \cdots & 2c_L & 2(c_L+1) \end{pmatrix} \in \mathbb{R}^{n_i\times n_i} \text{ if } i \neq l, k+1,$$

$$\Lambda_{ij} = \begin{pmatrix} 2c_L - 1 & \cdots & 2c_L - 1 \\ \vdots & \ddots & \vdots \\ 2c_L - 1 & \cdots & 2c_L - 1 \end{pmatrix} \in \mathbb{R}^{n_i \times n_j}.$$

For every $i \in \{1, \dots, k+1\}$, M_l has the following $n_i - 1$ eigenvectors:

$$\begin{aligned} w_{i1} &= (0, \dots, 0, |1, -1, 0 \dots, 0, 0, |0, \dots, 0), \\ w_{i2} &= (0, \dots, 0, |1, 0, -1 \dots, 0, 0, |0, \dots, 0), \\ &\vdots \\ w_{i\, n_i - 1} &= (0, \dots, 0, |\underbrace{1, 0, 0 \dots, 0, -1,}_{\Delta_i} |0, \dots, 0). \end{aligned}$$

We can verify this by multiplying the corresponding block Λ_{ii} with the Δ_i-part of the eigenvector w_{ij}. We have to consider the cases $i = l, i = k+1$ and $i \neq l, k+1$ separately.

For $i = l$, we find

$$M_l w_{lj} = \begin{pmatrix} 2c_L & \cdots & 2c_L \\ \vdots & \ddots & \vdots \\ 2c_L & \cdots & 2c_L \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \cdot w_{lj},$$

for $i = k+1$ we find

$$M_l w_{k+1\, j} = \begin{pmatrix} 2(c_L+1) & 2c_L - 1 & \cdots & 2c_L - 1 & 2c_L - 1 \\ 2c_L - 1 & 2(c_L+1) & \cdots & 2c_L - 1 & 2c_L - 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_L - 1 & 2c_L - 1 & \cdots & 2(c_L+1) & 2c_L - 1 \\ 2c_L - 1 & 2c_L - 1 & \cdots & 2c_L - 1 & 2(c_L+1) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2(c_L+1) - (2c_L - 1) \\ (2c_L - 1) - (2c_L - 1) \\ \vdots \\ (2c_L - 1) - (2c_L - 1) \\ (2c_L - 1) - 2(c_L + 1) \\ (2c_L - 1) - (2c_L - 1) \\ \vdots \\ (2c_L - 1) - (2c_L - 1) \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ \vdots \\ 0 \\ -3 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 3 \cdot w_{k+1\, j},$$

and for $i \neq l, k+1$ we find

$$M_l w_{ij} = \begin{pmatrix} 2(c_L+1) & 2c_L & \cdots & 2c_L & 2c_L \\ 2c_L & 2(c_L+1) & \cdots & 2c_L & 2c_L \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_L & 2c_L & \cdots & 2(c_L+1) & 2c_L \\ 2c_L & 2c_L & \cdots & 2c_L & 2(c_L+1) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2(c_L+1) - 2c_L \\ 2c_L - 2c_L \\ \vdots \\ 2c_L - 2c_L) \\ 2c_L - 2(c_L+1) \\ 2c_L - 2c_L \\ \vdots \\ 2c_L - 2c_L \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ \vdots \\ 0 \\ -2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 2 \cdot w_{ij}.$$

The eigenvalues are $0, 3, 2$ depending on whether $i = l, i = k+1$ or $i \neq l, k+1$ respectively. So in total we have $n - (k+1)$ eigenvectors of M_l with non-negative eigenvalues. The orthogonal complement of all these eigenvectors is spanned by

$$v_i = \frac{1}{n_i}(0, \dots, 0, |\underbrace{1, 1, \dots, 1,}_{\Delta_i}|0, \dots, 0), \quad i = 1, \dots, k+1,$$

since

$$\begin{aligned} \langle v_i, w_{ij} \rangle &= \frac{1}{n_i}(1 \cdot 1 + 1 \cdot 0 + \cdots + 1 \cdot 0 + 1 \cdot (-1) + 1 \cdot 0 + \cdots + 1 \cdot 0) = 0, \\ \langle v_i, w_{kj} \rangle &= 0, i \neq k, \\ \langle v_i, v_j \rangle &= 0. \end{aligned}$$

It is now sufficient to prove that the matrix $M_l' = (v_i M_l v_j^T)_{i,j=1,\dots,k+1} \in \mathbb{R}^{(k+1)\times(k+1)}$ is non-negative. We calculate its coefficients:

$$(M_l')_{ll} = v_l M_l v_l^T = \frac{1}{n_l^2}(1 \cdots 1) \begin{pmatrix} 2c_L & \cdots & 2c_L \\ \vdots & \ddots & \vdots \\ 2c_L & \cdots & 2c_L \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 2c_L,$$

$$
\begin{aligned}
&(M_l')_{k+1\,k+1} = v_{k+1} M_l v_{k+1}^T \\
&= \frac{1}{n_{k+1}^2}(1,\dots,1)\begin{pmatrix} 2(c_L+1) & 2c_L-1 & \cdots & 2c_L-1 & 2c_L-1 \\ 2c_L-1 & 2(c_L+1) & \cdots & 2c_L-1 & 2c_L-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_L-1 & 2c_L-1 & \cdots & 2(c_L+1) & 2c_L-1 \\ 2c_L-1 & 2c_L-1 & \cdots & 2c_L-1 & 2(c_L+1) \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\
&= \frac{1}{n_{k+1}}((n_{k+1}-1)(2c_L-1)+2(c_L+1)) = 2c_L - 1 + \frac{3}{n_{k+1}},
\end{aligned}
$$

$$
\begin{aligned}
&(M_l')_{ii} = v_i M_l v_i^T \\
&= \frac{1}{n_i^2}(1,\dots,1)\begin{pmatrix} 2(c_L+1) & 2c_L & \cdots & 2c_L & 2c_L \\ 2c_L & 2(c_L+1) & \cdots & 2c_L & 2c_L \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_L & 2c_L & \cdots & 2(c_L+1) & 2c_L \\ 2c_L & 2c_L & \cdots & 2c_L & 2(c_L+1) \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\
&= \frac{1}{n_i}((n_{k+1}-1)(2c_L)+2(c_L+1)) = 2\left(c_L + \frac{1}{n_i}\right),
\end{aligned}
$$

$$
(M_l')_{ij} = v_i M_l v_j^T = \frac{1}{n_i n_j}(1,\dots,1)\begin{pmatrix} 2c_L-1 & \cdots & 2c_L-1 \\ \vdots & \ddots & \vdots \\ 2c_L-1 & \cdots & 2c_L-1 \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 2c_L - 1.
$$

We investigate three cases.

Case 1: $2c_L = 1$. In this case, we have

- $(M_l')_{ll} = 1$,
- $(M_l')_{k+1\,k+1} = \frac{3}{n_{k+1}}$,
- $(M_l')_{ii} = 1 + \frac{2}{n_i}$,
- $(M_l')_{ij} = 0$,

so M_l' is a diagonal matrix with positive diagonal entries, so clearly it is positive semi-definite.

Case 2: $2c_L > 1$. We verify that the matrix $M_l'' = M_l'/(2c_L - 1)$ is positive semi-definite. Sylvester's criterion states that it is sufficient to verify that the $(j \times j)$-matrix in the upper left corner of M_l'' has positive determinant for all $j = 1, \dots, k+1$.

- $(M_l'')_{ll} = \frac{2c_L}{2c_L-1} = 1 + \frac{1}{2c_L-1}$,
- $(M_l'')_{k+1\,k+1} = \frac{2c_L-1+\frac{3}{n_{k+1}}}{2c_L-1} = 1 + \frac{3}{(n_{k+1})(2c_L-1)}$,
- $(M_l'')_{ii} = \frac{2\left(c_L+\frac{1}{n_i}\right)}{2c_L-1} = 1 + \frac{1+\frac{2}{n_i}}{2c_L-1}$,

- $(M_l'')_{ij} = 1$.

Since $(M_l'')_{ij} = 1$, we can apply lemma 7.2.2 to calculate the determinants the aforementioned submatrices. But all diagonal entries are strictly greater than 1, so these are clearly positive.

Case 3: $2c_L < 1$. We verify that the matrix $M_l'' = M_l'/(2c_L - 1)$ is negative semidefinite. Sylvester's criterion states that it is sufficient to verify that the $(j \times j)$-matrix in the upper left corner of M_l'' has a determinant with sign $(-1)^j$ for all $j = 1, \ldots, k+1$.

We get the same matrix elements as in Case 2, so we once again apply 7.2.2. We consider ranges of j:

If $1 \leq j < l$, we have

$$
\begin{aligned}
D_j &= \left(1 + \sum_{i=1}^{j} \frac{1}{\frac{1+\frac{2}{n_i}}{2c_L-1}}\right) \prod_{i=1}^{j} \left(\frac{1+\frac{2}{n_i}}{2c_L - 1}\right) = \left(1 + \sum_{i=1}^{j} \frac{2c_L - 1}{1 + \frac{2}{n_i}}\right) \prod_{i=1}^{j} \left(\frac{1+\frac{2}{n_i}}{2c_L - 1}\right) \\
&= \frac{1}{(2c_L - 1)^{j-1}} \left(\frac{1}{2c_L - 1} + \sum_{i=1}^{j} \frac{n_i}{n_i + 2}\right) \prod_{i=1}^{j} \left(1 + \frac{2}{n_i}\right),
\end{aligned}
$$

if $l \leq j < k+1$, we have

$$
\begin{aligned}
D_j &= \left(1 + \sum_{\substack{i=1 \\ i \neq l}}^{j} \frac{1}{\frac{1+\frac{2}{n_i}}{2c_L-1}} + \frac{1}{\frac{1}{2c_L-1}}\right) \prod_{\substack{i=1 \\ i \neq l}}^{j} \left(\frac{1+\frac{2}{n_i}}{2c_L - 1}\right) \left(\frac{1}{2c_L - 1}\right) \\
&= \left(2c_L + \sum_{\substack{i=1 \\ i \neq l}}^{j} \frac{2c_L - 1}{1 + \frac{2}{n_i}}\right) \prod_{\substack{i=1 \\ i \neq l}}^{j} \left(\frac{1+\frac{2}{n_i}}{2c_L - 1}\right) \left(\frac{1}{2c_L - 1}\right) \\
&= \frac{1}{(2c_L - 1)^{j-1}} \left(\frac{2c_L}{2c_L - 1} + \sum_{\substack{i=1 \\ i \neq l}}^{j} \frac{n_i}{n_i + 2}\right) \prod_{\substack{i=1 \\ i \neq l}}^{j} \left(1 + \frac{2}{n_i}\right),
\end{aligned}
$$

and finally if $j = k+1$, we have

$$\begin{aligned}
D_{k+1} = &\left(1 + \sum_{\substack{i=1\\ i\neq l}}^{k} \frac{1}{\frac{1+\frac{2}{n_i}}{2c_L-1}} + \frac{1}{\frac{1}{2c_L-1}} + \frac{1}{\frac{3}{n_{k+1}(2c_L-1)}}\right)\\
&\cdot \prod_{\substack{i=1\\ i\neq l}}^{k} \left(\frac{1+\frac{2}{n_i}}{2c_L-1}\right)\left(\frac{1}{2c_L-1}\right)\left(\frac{3}{n_{k+1}(2c_L-1)}\right)\\
= &\left(2c_L + \sum_{\substack{i=1\\ i\neq l}}^{k} \frac{2c_L-1}{1+\frac{2}{n_i}} + \frac{n_{k+1}(2c_L-1)}{3}\right)\\
&\cdot \prod_{\substack{i=1\\ i\neq l}}^{k} \left(\frac{1+\frac{2}{n_i}}{2c_L-1}\right)\left(\frac{1}{2c_L-1}\right)\left(\frac{3}{n_{k+1}(2c_L-1)}\right)\\
= &\frac{1}{(2c_L-1)^k}\left(\frac{2c_L}{2c_L-1} + \sum_{\substack{i=1\\ i\neq l}}^{k} \frac{n_i}{n_i+2} + \frac{n_{k+1}}{3}\right)\prod_{\substack{i=1\\ i\neq l}}^{k}\left(1+\frac{2}{n_i}\right)\left(\frac{3}{n_{k+1}}\right).
\end{aligned}$$

Hence we have:

$$\begin{aligned}
\operatorname{sign}(D_j) &= (-1)^{j-1}\operatorname{sign}\left(\frac{1}{2c_L-1} + \sum_{i=1}^{j}\frac{n_i}{n_i+2}\right) && 1 \leq j < l,\\
\operatorname{sign}(D_j) &= (-1)^{j-1}\operatorname{sign}\left(\frac{2c_L}{2c_L-1} + \sum_{\substack{i=1\\ i\neq l}}^{j}\frac{n_i}{n_i+2}\right) && l \leq j \leq k,\\
\operatorname{sign}(D_{k+1}) &= (-1)^{k}\operatorname{sign}\left(\frac{2c_L}{2c_L-1} + \sum_{\substack{i=1\\ i\neq l}}^{k}\frac{n_i}{n_i+2} + \frac{n_{k+1}}{3}\right).
\end{aligned}$$

So we always need the formula inside the sign-function on the right-hand sides of the previous equations to be negative:

$$\begin{cases}
\frac{1}{2c_L-1} + \sum_{i=1}^{j}\frac{n_i}{n_i+2} \leq 0 & 1 \leq j < l,\\
\frac{2c_L}{2c_L-1} + \sum_{\substack{i=1\\ i\neq l}}^{j}\frac{n_i}{n_i+2} \leq 0 & l \leq j \leq k,\\
\frac{2c_L}{2c_L-1} + \sum_{\substack{i=1\\ i\neq l}}^{k}\frac{n_i}{n_i+2} + \frac{n_{k+1}}{3} \leq 0.
\end{cases}$$

But since we are in the case where $2c_L < 1$, we know that $2c_L/(2c_L-1) > 1/(2c_L-1)$. Moreover, it is obvious that for all $1 \leq i \leq k$, $n_i/(n_i+2) \geq 0$ and $n_{k+1}/3 \geq 0$. So the last condition implies the first k conditions.

Now, note that

$$\sum_{\substack{i=1\\ i\neq l}}^{k}\frac{n_i}{n_i+2} = \sum_{\substack{i=1\\ i\neq l}}^{k}\frac{n_i+2}{n_i+2} - \sum_{\substack{i=1\\ i\neq l}}^{k}\frac{2}{n_i+2} = k-1-2\sum_{\substack{i=1\\ i\neq l}}^{k}\frac{1}{n_i+2}.$$

Then applying this and multiplying the condition by $2c_L - 1$ gives

$$2c_L + (2c_L - 1)\left(k - 1 - 2\sum_{\substack{i=1\\ i\neq l}}^{k} \frac{1}{n_i+2} + \frac{n_{k+1}}{3}\right) \leq 0.$$

Solving for $2c_L$ gives

$$2c_L \geq \frac{n_{k+1} + 3k - 3 - 6\sum_{\substack{i=1\\ i\neq l}}^{k} \frac{1}{n_i+2}}{n_{k+1} + 3k - 6\sum_{\substack{i=1\\ i\neq l}}^{k} \frac{1}{n_i+2}}. \tag{7.2.8}$$

Step 3: Finding the best possible c_L in (II).

We use that

$$\begin{aligned}\sum_b C_{tbb} &= \sum_i \sum_{\alpha_i} C_{\alpha_i\alpha_i t}^2 + C_{ttt}^2 + 2\sum_{i<j}\sum_{\alpha_i,\alpha_j} C_{\alpha_i\alpha_i t}C_{\alpha_j\alpha_j t} \\ &+ 2\sum_i \sum_{\alpha_i<\beta_i} C_{\alpha_i\alpha_i t}C_{\beta_i\beta_i t} + 2\sum_i\sum_{\alpha_i,r} C_{\alpha_i\alpha_i t}C_{rrt} + 2\sum_{r<s} C_{rrt}C_{sst} + \sum_r C_{rrt}^2.\end{aligned}$$

So we can rearrange (II) into

$$\begin{aligned}&(c_L+1)\sum_i\sum_{\alpha_i} C_{\alpha_i\alpha_i t}^2 + c_L C_{ttt}^2 + (c_L+1)\sum_{r\neq t} C_{rrt}^2 + 2c_L \sum_i\sum_{\alpha_i<\beta_i} C_{\alpha_i\alpha_i t}C_{\beta_i\beta_i t} \\ &+ (2c_L-1)\left(\sum_{i<j}\sum_{\alpha_i,\alpha_j} C_{\alpha_i\alpha_i t}C_{\alpha_j\alpha_j t} + \sum_i\sum_{\alpha_i,r} C_{\alpha_i\alpha_i t}C_{rrt} + \sum_{r<s} C_{rrt}C_{sst}\right) \qquad \geq 0.\end{aligned}$$

Now, if we put $x_a = C_{aat}$ for all $a = 1, \dots, n$, we can consider the left-hand side of the previous inequality as a quadratic form on $\mathbb{R}^n$. So we need to find necessary and sufficient conditions on c_L for this quadratic form to be non-negative. Two times the matrix of this quadratic form consists of $(k+1)^2$ blocks:

$$M_t = (\Lambda_{ij})_{i,j=1,\dots,k+1}$$

with

$$\Lambda_{ii} = \begin{pmatrix} 2(c_L+1) & 2c_L & \cdots & 2c_L & 2c_L \\ 2c_L & 2(c_L+1) & \cdots & 2c_L & 2c_L \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_L & 2c_L & \cdots & 2(c_L+1) & 2c_L \\ 2c_L & 2c_L & \cdots & 2c_L & 2(c_L+1) \end{pmatrix} \in \mathbb{R}^{n_i\times n_i} \text{ if } i \neq k+1,$$

$$\Lambda_{ij} = \begin{pmatrix} 2c_L - 1 & \cdots & 2c_L - 1 \\ \vdots & \ddots & \vdots \\ 2c_L - 1 & \cdots & 2c_L - 1 \end{pmatrix} \in \mathbb{R}^{n_i\times n_j},$$

$$\Lambda_{k+1\,k+1} = \begin{pmatrix} 2(c_L+1) & 2c_L-1 & \cdots & 2c_L-1 & 2c_L-1 \\ 2c_L-1 & 2(c_L+1) & \cdots & 2c_L-1 & 2c_L-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_L-1 & 2c_L-1 & \cdots & 2(c_L+1) & 2c_L-1 \\ 2c_L-1 & 2c_L-1 & \cdots & 2c_L-1 & 2c_L \end{pmatrix} \in \mathbb{R}^{n_{k+1}\times n_{k+1}},$$

where we assumed without loss of generality that $t = n$.

For every $i \in \{1,\dots,k\}$, M_t has the following $n_i - 1$ eigenvectors:

$$\begin{aligned} w_{i1} &= (0,\dots,0,|1,-1,0\dots,0,0,|0,\dots,0), \\ w_{i2} &= (0,\dots,0,|1,0,-1\dots,0,0,|0,\dots,0), \\ &\vdots \\ w_{i\,n_i-1} &= (0,\dots,0,|\underbrace{1,0,0\dots,0,-1,}_{\Delta_i}|0,\dots,0), \end{aligned}$$

and M_t has another $n_{k+1}-2$ eigenvectors

$$\begin{aligned} w_{k+1\,1} &= (0,\dots,0,|1,-1,0\dots,0,0,0), \\ w_{k+1\,2} &= (0,\dots,0,|1,0,-1\dots,0,0,0), \\ &\vdots \\ w_{k+1\,n_{k+1}-2} &= (0,\dots,0,|\underbrace{1,0,0\dots,0,-1,0}_{\Delta_{k+1}}). \end{aligned}$$

The eigenvalues of w_{ij} are again $0,3,2$ depending on whether $i = l, i = k+1$ or $i \neq l, k+1$ respectively. So in total we have $n-(k+2)$ eigenvectors of M_t with non-negative eigenvalues. The orthogonal complement of all these eigenvectors is spanned by

$$\begin{aligned} v_i &= \frac{1}{n_i}(0,\dots,0,|\underbrace{1,1,\dots,1,}_{\Delta_i}|0,\dots,0), \quad i=1,\dots,k, \\ v_{k+1} &= \frac{1}{n_{k+1}-1}(0,\dots,0,|\underbrace{1,1,\dots,1,0}_{\Delta_{k+1}}), \\ v_{k+2} &= (0,\dots,0,1). \end{aligned}$$

It is now sufficient to prove that the matrix $M_t' = (v_i M_t v_j^T)_{i,j=1,\dots,k+2} \in \mathbb{R}^{(k+2)\times(k+2)}$ is non-negative.

$$\begin{aligned} (M_t')_{k+1\,k+1} &= v_{k+1}M_t v_{k+1}^T \\ &= \frac{1}{(n_{k+1}-1)^2}(1,\dots,1)\begin{pmatrix} 2(c_L+1) & 2c_L-1 & \cdots & 2c_L-1 & 2c_L-1 \\ 2c_L-1 & 2(c_L+1) & \cdots & 2c_L-1 & 2c_L-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_L-1 & 2c_L-1 & \cdots & 2(c_L+1) & 2c_L-1 \\ 2c_L-1 & 2c_L-1 & \cdots & 2c_L-1 & 2(c_L+1) \end{pmatrix}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} \\ &= \frac{1}{n_{k+1}-1}((n_{k+1}-2)(2c_L-1)+2(c_L+1)) = 2c_L - 1 + \frac{3}{n_{k+1}-1}, \end{aligned}$$

$$
\begin{aligned}
(M_t')_{ii} &= v_i M_t v_i^T \\
&= \frac{1}{n_i^2}(1,\dots,1)\begin{pmatrix} 2(c_L+1) & 2c_L & \cdots & 2c_L & 2c_L \\ 2c_L & 2(c_L+1) & \cdots & 2c_L & 2c_L \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_L & 2c_L & \cdots & 2(c_L+1) & 2c_L \\ 2c_L & 2c_L & \cdots & 2c_L & 2(c_L+1) \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\
&= \frac{1}{n_i}((n_{k+1}-1)(2c_L)+2(c_L+1)) = 2\left(c_L+\frac{1}{n_i}\right),
\end{aligned}
$$

$$
(M_t')_{ij} = v_i M_t v_j^T = \frac{1}{n_i n_j}(1,\dots,1)\begin{pmatrix} 2c_L-1 & \cdots & 2c_L-1 \\ \vdots & \ddots & \vdots \\ 2c_L-1 & \cdots & 2c_L-1 \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 2c_L - 1,
$$

$$
(M_t')_{k+2\ k+2} = v_{k+2} M_t v_{k+2}^T = 2c_L.
$$

We investigate three cases.

Case 1: $2c_L = 1$. In this case, we have

- $(M_t')_{k+1\ k+1} = \frac{3}{n_{k+1}-1}$,
- $(M_t')_{ii} = 1 + \frac{2}{n_i}$,
- $(M_t')_{ij} = 0$,
- $(M_t')_{k+2\ k+2} = 1$,

so M_t' is a diagonal matrix with positive diagonal entries, so clearly it is positive semi-definite.

Case 2: $2c_L > 1$. We verify that the matrix $M_t'' = M_t'/(2c_L-1)$ is positive semi-definite. Sylvester's criterion states that it is sufficient to verify that the $(j \times j)$-matrix in the upper left corner of M_t'' has positive determinant for all $j = 1, \dots, k+2$.

- $(M_t'')_{k+1\ k+1} = \frac{2c_L - 1 + \frac{3}{n_{k+1}-1}}{2c_L-1} = 1 + \frac{3}{(n_{k+1}-1)(2c_L-1)}$,
- $(M_t'')_{ii} = \frac{2\left(c_L+\frac{1}{n_i}\right)}{2c_L-1} = 1 + \frac{1+\frac{2}{n_i}}{2c_L-1}$,
- $(M_t'')_{ij} = 1$,
- $(M_t'')_{k+2\ k+2} = \frac{2c_L}{2c_L-1} = 1 + \frac{1}{2c_L-1}$.

Since $(M_t'')_{ij} = 1$, we can apply lemma 7.2.2 to calculate the determinants the aforementioned submatrices. But all diagonal entries are strictly greater than 1, so these are clearly positive.

Case 3: $2c_L < 1$. We verify that the matrix $M_t'' = M_t'/(2c_L - 1)$ is negative semi-definite. Sylvester's criterion states that it is sufficient to verify that the $(j \times j)$-matrix in the upper left corner of M_t'' has a determinant with sign $(-1)^j$ for all $j = 1, \dots, k+2$.

We get the same matrix elements as in Case 2, so we once again apply 7.2.2. We consider ranges of j:

If $1 \leq j \leq k$, we have

$$\begin{aligned} D_j &= \left(1 + \sum_{i=1}^{j} \frac{1}{\frac{1+\frac{2}{n_i}}{2c_L-1}}\right) \prod_{i=1}^{j} \left(\frac{1+\frac{2}{n_i}}{2c_L - 1}\right) = \left(1 + \sum_{i=1}^{j} \frac{2c_L - 1}{1+\frac{2}{n_i}}\right) \prod_{i=1}^{j} \left(\frac{1+\frac{2}{n_i}}{2c_L - 1}\right) \\ &= \frac{1}{(2c_L-1)^{j-1}} \left(\frac{1}{2c_L - 1} + \sum_{i=1}^{j} \frac{n_i}{n_i + 2}\right) \prod_{i=1}^{j} \left(1 + \frac{2}{n_i}\right), \end{aligned}$$

if $j = k+1$, we have

$$\begin{aligned} D_{k+1} &= \left(1 + \sum_{i=1}^{k} \frac{1}{\frac{1+\frac{2}{n_i}}{2c_L-1}} + \frac{1}{\frac{3}{(n_{k+1}-1)(2c_L-1)}}\right) \prod_{i=1}^{k} \left(\frac{1+\frac{2}{n_i}}{2c_L - 1}\right) \left(\frac{3}{(n_{k+1}-1)(2c_L-1)}\right) \\ &= \left(1 + \sum_{i=1}^{k} \frac{2c_L - 1}{1+\frac{2}{n_i}} + \frac{(n_{k+1}-1)(2c_L-1)}{3}\right) \prod_{i=1}^{k} \left(\frac{1+\frac{2}{n_i}}{2c_L - 1}\right) \left(\frac{3}{(n_{k+1}-1)(2c_L-1)}\right) \\ &= \frac{1}{(2c_L-1)^{k}} \left(\frac{1}{2c_L - 1} + \sum_{i=1}^{k} \frac{n_i}{n_i + 2} + \frac{n_{k+1}-1}{3}\right) \prod_{i=1}^{k} \left(1 + \frac{2}{n_i}\right) \left(\frac{3}{n_{k+1}-1}\right), \end{aligned}$$

and finally if $j = k+2$, we have

$$\begin{aligned} D_{k+2} = &\left(1 + \sum_{i=1}^{k} \frac{1}{\frac{1+\frac{2}{n_i}}{2c_L-1}} + \frac{1}{\frac{3}{(n_{k+1}-1)(2c_L-1)}} + \frac{1}{\frac{1}{2c_L-1}}\right) \\ &\cdot \prod_{i=1}^{k} \left(\frac{1+\frac{2}{n_i}}{2c_L - 1}\right) \left(\frac{3}{(n_{k+1}-1)(2c_L-1)}\right) \left(\frac{1}{2c_L-1}\right) \\ = &\left(2c_L + \sum_{i=1}^{k} \frac{2c_L - 1}{1+\frac{2}{n_i}} + \frac{(n_{k+1}-1)(2c_L-1)}{3}\right) \\ &\cdot \prod_{i=1}^{k} \left(\frac{1+\frac{2}{n_i}}{2c_L - 1}\right) \left(\frac{3}{(n_{k+1}-1)(2c_L-1)}\right) \left(\frac{1}{2c_L-1}\right) \\ = &\frac{1}{(2c_L-1)^{k+1}} \left(\frac{2c_L}{2c_L - 1} + \sum_{i=1}^{k} \frac{n_i}{n_i + 2} + \frac{n_{k+1}-1}{3}\right) \prod_{i=1}^{k} \left(1 + \frac{2}{n_i}\right) \left(\frac{3}{n_{k+1}-1}\right). \end{aligned}$$

Hence we have

$$\begin{aligned}
\operatorname{sign}(D_j) &= (-1)^{j-1} \operatorname{sign}\left(\frac{1}{2c_L - 1} + \sum_{i=1}^{j} \frac{n_i}{n_i + 2}\right) & 1 \le j \le k,\\
\operatorname{sign}(D_{k+1}) &= (-1)^{k} \operatorname{sign}\left(\frac{1}{2c_L - 1} + \sum_{i=1}^{k} \frac{n_i}{n_i + 2} + \frac{n_{k+1} - 1}{3}\right), \\
\operatorname{sign}(D_{k+2}) &= (-1)^{k+1} \operatorname{sign}\left(\frac{2c_L}{2c_L - 1} + \sum_{i=1}^{k} \frac{n_i}{n_i + 2} + \frac{n_{k+1} - 1}{3}\right).
\end{aligned}$$

So we always need the formula inside the sign-function on the right-hand sides of the previous equations to be negative:

$$\begin{cases}
\frac{1}{2c_L-1} + \sum_{i=1}^{j} \frac{n_i}{n_i+2} \le 0 & 1 \le j < k+1, \\
\frac{1}{2c_L-1} + \sum_{i=1}^{k} \frac{n_i}{n_i+2} + \frac{n_{k+1}-1}{3} \le 0 & , \\
\frac{2c_L}{2c_L-1} + \sum_{i=1}^{k} \frac{n_i}{n_i+2} + \frac{n_{k+1}-1}{3} \le 0. &
\end{cases}$$

But since we are in the case where $2c_L < 1$, we know that $2c_L/(2c_L - 1) > 1/(2c_L - 1)$. Moreover, it is obvious that for all i, $n_i/(n_i + 2) \ge 0$ and $(n_{k+1} - 1)/3 \ge 0$. So the last condition implies the first $k + 1$ conditions.

Now, note that

$$\sum_{i=1}^{k} \frac{n_i}{n_i + 2} = \sum_{i=1}^{k} \frac{n_i + 2}{n_i + 2} - \sum_{i=1}^{k} \frac{2}{n_i + 2} = k - 2\sum_{i=1}^{k} \frac{1}{n_i + 2},$$

Then applying this and multiplying the condition by $2c_L - 1$ gives

$$2c_L + (2c_L - 1)\left(k - 2\sum_{i=1}^{k} \frac{1}{n_i + 2} + \frac{n_{k+1} - 1}{3}\right) \le 0,$$

Solving for $2c_L$ gives

$$2c_L \ge \frac{n_{k+1} + 3k - 1 - 6\sum_{i=1}^{k} \frac{1}{n_i+2}}{n_{k+1} + 3k + 2 - 6\sum_{i=1}^{k} \frac{1}{n_i+2}}.$$

We show that the right-hand side of the c_L found in step 2 is less than the right-hand side of step 3. We define

$$\lambda = n_{k+1} + 3k - 6\sum_{\substack{i=1 \\ i \ne l}}^{k} \frac{1}{n_i + 2},$$

Thus we have to show that

$$\frac{\lambda - 3}{\lambda} \le \frac{\lambda - 1 - \frac{6}{n_l+2}}{\lambda + 2 - \frac{6}{n_l+2}}.$$

which is true for $n_l \ge 1$. But by definition, $n_l \ge 2$, thus we choose for c_L the value found in step 3.

Step 4: The equality case. Assume equality holds at a point. Then we have equality in (7.2.6), which gives us condition (i) of the equality case. Next, we have equality in statement (I), this implies that the vector $(C_{11\alpha_i}, \dots, C_{nn\alpha_i})$ has to be a linear combination of the eigenvectors w_{ij} of M_l. This gives condition (ii) of the equality case. Similarly we obtain from equality in statement (II) condition (iii) of the equality case. $\square$

Theorem 7.2.4. *Let M^n be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4\tilde{c})$. Let $(n_1, \dots, n_k) \in \mathcal{S}(n)$ satisfying $n_1 + \dots + n_k = n$. Then, at any point of M^n, we have*

$$\delta(n_1, \dots, n_k) \leq \frac{n^2\left(k - 1 - 1\sum_{i=2}^{k} \frac{1}{2+n_i}\right)}{2\left(k - 2\sum_{i=1}^{k} \frac{1}{2+n_i}\right)} \|H\|^2 + b(n_1, \dots, n_k)\tilde{c}. \tag{7.2.9}$$

Assume that equality holds at a point $p \in M^n$. Then with the choice of basis and the notations introduced earlier in this chapter, one has

(i) $C_{a\alpha_i\alpha_j} = 0$ if $i \neq j$ and $a \neq \alpha_i, \alpha_j$,

(ii) if $n_j \neq n_1$: $C_{\alpha_i\alpha_i\beta_j} = 0$ if $i \neq j$ and $\sum_{\alpha_j \in \Delta_j} C_{\alpha_j\alpha_j\beta_j} = 0$,

(iii) if $n_j = n_1$: $\sum_{\alpha_j \in \Delta_j} C_{\alpha_j\alpha_j\beta_j} = (n_i + 2)C_{\alpha_i\alpha_i\beta_j}$ for any $i \neq j$ and any $\alpha_i \in \Delta_i$.

Proof. The proof consists of four steps.

Step 1: Set-up. The set-up of the proof is exactly the same as the previous theorem, except that now $\Delta_{k+1} = \emptyset$. So we have

$$\tau - \sum_i \tau(L_i) = \sum_a \sum_{i<j} \sum_{\alpha_i, \alpha_j} (C_{a\alpha_i\alpha_i} C_{a\alpha_j\alpha_j} - C_{a\alpha_i\alpha_j}^2) + b(n_1, \dots, n_k)\tilde{c}. \tag{7.2.10}$$

Now, let us consider the quadratic terms in the summations:

$$\sum_a \sum_{i<j} \sum_{\alpha_i, \alpha_j} C_{a\alpha_i\alpha_j}^2 \geq \sum_i \sum_{\alpha_i} \sum_{b \notin \Delta_i} C_{b\alpha_i\alpha_i}^2, \tag{7.2.11}$$

since every term on the right-hand side is also a term on the left-hand side, but not vice-versa. We then find by combining (7.2.10) and (7.2.11) that

$$\tau - \sum_i \tau(L_i) \leq \sum_a \left(\sum_{i<j} \sum_{\alpha_i, \alpha_j} C_{a\alpha_i\alpha_i} C_{a\alpha_j\alpha_j} \right) - \sum_i \sum_{\alpha_i} \sum_{b \notin \Delta_i} C_{b\alpha_i\alpha_i}^2 + b(n_1, \dots, n_k)\tilde{c}. \tag{7.2.12}$$

Now, we want to find the "best" c_L such that (7.2.12) is less than or equal to

$$n^2 c_L(n_1, \dots, n_k)\|H\|^2 + b(n_1, \dots, n_k)\tilde{c} = c_L(n_1, \dots, n_k) \sum_a \left(\sum_b C_{abb} \right)^2 + b(n_1, \dots, n_k)\tilde{c}.$$

In fact, we want to prove that the value for c_L is the best possible one in the sense that the inequality in the theorem will no longer be true in general for smaller values of c_L. In view of Lemma 1, we have to find the smallest possible c_L for which the following statement holds:

for any $l \in \{1, \dots, k\}$ and any $\gamma_l \in \Delta_l$:

$$\sum_{i<j} \sum_{\alpha_i,\alpha_j} C_{\gamma_l\alpha_i\alpha_i} C_{\gamma_l\alpha_j\alpha_j} - \sum_{i\neq l} \sum_{\alpha_i} C^2_{\gamma_l\alpha_i\alpha_i} \leq c_L(n_1, \dots, n_k) \left(\sum_b C_{\gamma_l bb} \right)^2 . \tag{7.2.13}$$

Step 2: Finding the best possible c_L.

We use that

$$\begin{aligned}\sum_b C_{\gamma_l bb} = &\sum_{i\neq l} \sum_{\alpha_i} C^2_{\alpha_i\alpha_i\gamma_l} + \sum_{\alpha_l} C^2_{\alpha_l\alpha_l\gamma_l} + 2\sum_{i<j} \sum_{\alpha_i,\alpha_j} C_{\alpha_i\alpha_i\gamma_l} C_{\alpha_j\alpha_j\gamma_l} \\ &+ 2\sum_i \sum_{\alpha_i<\beta_i} C_{\alpha_i\alpha_i\gamma_l} C_{\beta_i\beta_i\gamma_l}.\end{aligned}$$

So we can rearrange (7.2.13) into

$$\begin{aligned}&(c_L+1) \sum_{i\neq l} \sum_{\alpha_i} C^2_{\alpha_i\alpha_i\gamma_l} + c_L \sum_{\alpha_l} C^2_{\alpha_l\alpha_l\gamma_l} + 2c_L \sum_i \sum_{\alpha_i<\beta_i} C_{\alpha_i\alpha_i\gamma_l} C_{\beta_i\beta_i\gamma_l} \\ &+ (2c_L - 1) \left(\sum_{i<j} \sum_{\alpha_i,\alpha_j} C_{\alpha_i\alpha_i\gamma_l} C_{\alpha_j\alpha_j\gamma_l} \right) \geq 0.\end{aligned}$$

Now, if we put $x_a = C_{aa\gamma_l}$ for all $a = 1, \dots, n$, we can consider the left-hand side of the previous inequality as a quadratic form on $\mathbb{R}^n$. So we need to find necessary and sufficient conditions on c_L for this quadratic form to be non-negative. Two times the matrix of this quadratic form consists of k^2 blocks:

$$M_l = (\Lambda_{ij})_{i,j=1,\dots,k},$$

with

$$\Lambda_{ll} = \begin{pmatrix} 2c_L & \cdots & 2c_L \\ \vdots & \ddots & \vdots \\ 2c_L & \cdots & 2c_L \end{pmatrix} \in \mathbb{R}^{n_l\times n_l},$$

$$\Lambda_{ii} = \begin{pmatrix} 2(c_L+1) & 2c_L & \cdots & 2c_L & 2c_L \\ 2c_L & 2(c_L+1) & \cdots & 2c_L & 2c_L \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2c_L & 2c_L & \cdots & 2(c_L+1) & 2c_L \\ 2c_L & 2c_L & \cdots & 2c_L & 2(c_L+1) \end{pmatrix} \in \mathbb{R}^{n_i\times n_i} \text{ if } i \neq l,$$

$$\Lambda_{ij} = \begin{pmatrix} 2c_L - 1 & \cdots & 2c_L - 1 \\ \vdots & \ddots & \vdots \\ 2c_L - 1 & \cdots & 2c_L - 1 \end{pmatrix} \in \mathbb{R}^{n_i\times n_j}.$$

The next steps are identical to Step 2 from the previous theorem once again, so we omit some details. We take the same eigenvectors v_i. It is now sufficient to prove that the matrix $M_l' = (v_i M_l v_j^T)_{i,j=1,\dots,k} \in \mathbb{R}^{(k)\times(k)}$ is non-negative. This matrix has the following components:

$$\begin{aligned}
(M_l')_{ll} &= 2c_L, \\
(M_l')_{ii} &= 2\left(c_L + \frac{1}{n_i}\right) \quad i \neq l, \\
(M_l')_{ij} &= 2c_L - 1.
\end{aligned}$$

We investigate three cases.

Case 1: $2c_L = 1$. In this case, we have

- $(M_l')_{ll} = 1$,
- $(M_l')_{ii} = 1 + \frac{2}{n_i}$,
- $(M_l')_{ij} = 0$.

so M_l' is a diagonal matrix with positive diagonal entries, so clearly it is positive semi-definite.

Case 2: $2c_L > 1$. We verify that the matrix $M_l'' = M_l'/(2c_L - 1)$ is positive semi-definite. Sylvester's criterion states that it is sufficient to verify that the $(j \times j)$-matrix in the upper left corner of M_l'' has positive determinant for all $j = 1, \dots, k$.

- $(M_l'')_{ll} = \frac{2c_L}{2c_L - 1} = 1 + \frac{1}{2c_L - 1}$,
- $(M_l'')_{ii} = \frac{2\left(c_L + \frac{1}{n_i}\right)}{2c_L - 1} = 1 + \frac{1 + \frac{2}{n_i}}{2c_L - 1}$,
- $(M_l'')_{ij} = 1$.

Since $(M_l'')_{ij} = 1$, we can apply lemma 7.2.2 to calculate the determinants the aforementioned submatrices. But all diagonal entries are strictly greater than 1, so these are clearly positive.

Case 3: $2c_L < 1$. We verify that the matrix $M_l'' = M_l'/(2c_L - 1)$ is negative semi-definite. Sylvester's criterion states that it is sufficient to verify that the $(j \times j)$-matrix in the upper left corner of M_l'' has a determinant with sign $(-1)^j$ for all $j = 1, \dots, k$.

We get the same matrix elements as in Case 2, so we once again apply 7.2.2. We consider ranges of j:

If $1 \leq j < l$, we have

$$D_j = \frac{1}{(2c_L - 1)^{j-1}} \left(\frac{1}{2c_L - 1} + \sum_{i=1}^{j} \frac{n_i}{n_i + 2} \right) \prod_{i=1}^{j} \left(1 + \frac{2}{n_i}\right),$$

and if $l \leq j \leq k$, we have

$$D_j = \frac{1}{(2c_L - 1)^{j-1}} \left(\frac{2c_L}{2c_L - 1} + \sum_{\substack{i=1 \\ i \neq l}}^{j} \frac{n_i}{n_i + 2} \right) \prod_{\substack{i=1 \\ i \neq l}}^{j} \left(1 + \frac{2}{n_i}\right).$$

Hence we have

$$\operatorname{sign}(D_j) = (-1)^{j-1} \operatorname{sign}\left(\frac{1}{2c_L - 1} + \sum_{i=1}^{j} \frac{n_i}{n_i + 2}\right) \qquad 1 \le j < l,$$
$$\operatorname{sign}(D_j) = (-1)^{j-1} \operatorname{sign}\left(\frac{2c_L}{2c_L - 1} + \sum_{\substack{i=1 \\ i \neq l}}^{j} \frac{n_i}{n_i + 2}\right) \qquad l \le j \le k.$$

So we always need the formula inside the sign-function on the right-hand sides of the previous equations to be negative:

$$\begin{cases} \frac{1}{2c_L-1} + \sum_{i=1}^{j} \frac{n_i}{n_i+2} \le 0 & 1 \le j < l, \\ \frac{2c_L}{2c_L-1} + \sum_{\substack{i=1 \\ i\neq l}}^{j} \frac{n_i}{n_i+2} \le 0 & l \le j \le k. \end{cases}$$

But since we are in the case where $2c_L < 1$, we know that $2c_L/(2c_L - 1) > 1/(2c_L - 1)$. Moreover, it is obvious that for all i, $n_i/(n_i + 2) \ge 0$. So the last condition with $j = k$ implies the first $k - 1$ conditions. Moreover, since $n_1 \le n_2 \le \cdots \le n_k$, the k-th condition with $l = 1$ is the strongest:

$$\frac{2c_L}{2c_L - 1} + \sum_{i=2}^{k} \frac{n_i}{n_i + 2} = \frac{2c_L}{2c_L - 1} + k - 1 - 2\sum_{i=2}^{k} \frac{1}{n_i + 2} \le 0$$

Multiplying the condition by $2c_L - 1$ gives

$$2c_L + (2c_L - 1)\left(k - 1 - 2\sum_{i=2}^{k} \frac{1}{n_i + 2}\right) \le 0,$$

and solving for $2c_L$ gives

$$2c_L \ge \frac{k - 1 - 2\sum_{i=2}^{k} \frac{1}{n_i+2}}{k - 2\sum_{i=2}^{k} \frac{1}{n_i+2}}.$$

Step 3: The equality case. Assume equality holds at a point. Then we have equality in (7.2.12), which gives us condition (i) of the equality case. Next, we have equality in (7.2.13), this implies that the vector $(C_{11\gamma_l}, \ldots, C_{nn\gamma_l})$ has to be in the kernel of M_l. If $n_l \neq n_1$ then this kernel is spanned by the eigenvectors w_{lj}. This corresponds to condition (ii) of the equality case. If $n_l = n_1$, the kernel of M_l is larger due to the choice of c_L. In particular, there will be non-zero linear combinations of the vectors $v_1, \ldots, v_k$ in the kernel. Assume that

$$M_l\left(\sum_{i=1}^{k} a_i v_i\right) = 0, \tag{7.2.14}$$

for some real numbers $\{a_1, \ldots, a_k\}$.

We calculate $M_l v_i$. If $i \neq l$, we find

$$M_l v_i = \frac{1}{n_i} \begin{pmatrix} \Lambda_{i1} \\ \vdots \\ \Lambda_{ii} \\ \vdots \\ \Lambda_{ik} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 2c_L - 1 \\ \vdots \\ 2c_L - 1 + \left(1 + \frac{2}{n_i}\right) \\ \vdots \\ 2c_L - 1 \end{pmatrix},$$

and for $i = l$ we obtain

$$M_l v_l = \frac{1}{n_l} \begin{pmatrix} \Lambda_{l1} \\ \vdots \\ \Lambda_{ll} \\ \vdots \\ \Lambda_{lk} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 2c_L - 1 \\ \vdots \\ 2c_L \\ \vdots \\ 2c_L - 1 \end{pmatrix}.$$

As such, we find that (7.2.14) is equivalent to

$$\begin{cases} \left(\sum_{j=1}^k a_j\right)(2c_L - 1) + a_i \left(1 + \frac{2}{n_i}\right) = 0, & 1 \leq i \leq k, i \neq l, \\ \left(\sum_{j=1}^k a_j\right)(2c_L - 1) + a_l = 0. \end{cases}$$

From this we determine that

$$a_i = a_l \left(\frac{n_i}{n_i + 2}\right),$$

for all $i \neq l$. But then we can calculate the sum of the a_i and solve the conditions for c_L:

$$\left(\sum_{j=1}^k a_j\right)(2c_L - 1) + a_l = \left(\sum_{i \neq l} a_l \left(\frac{n_i}{n_i + 2}\right) + a_l\right)(2c_L - 1) + a_l = 0.$$

Because the solution cannot be zero, $a_l \neq 0$ and we may divide by a_l:

$$\left(\sum_{i \neq l} \frac{n_i}{n_i + 2} + 1\right)(2c_L - 1) + 1 = 0.$$

Solving this for $2c_L$ and rewriting the summation gives us

$$2c_L = \frac{k - 1 - 2\sum_{i \neq l} \frac{1}{n_i + 2}}{k - 2\sum_{i \neq l} \frac{1}{n_i + 2}},$$

from which is clear that we have a solution if and only if $n_l = n_1$. Thus the vector $(C_{11\gamma_l}, \ldots, C_{nn\gamma_l})$ has to satisfy condition (iii) of the equality case. □

Definition 7.2.5. Let $\phi : M^n \to \tilde{M}^n(4\tilde{c})$ be an isometric immersion of a Lagrangian submanifold in a complex space form. If this immersion satisfies the equality in (7.2.3) or (7.2.9) for some $(n_1, \ldots, n_k)$ at every point $p \in M$, then it is called an *improved ideal immersion*.

Chapter 8

Corollaries

In this chapter, we give some corollaries of the improved inequality. In the first section we will give a non-immersibility theorem for compact Lagrangian submanifolds, in the second section we give corollaries related to k-tuples where $k = 0$ or $k = 1$.

8.1 Non-immersibility of compact manifolds

Definition 8.1.1. Let M^n be a Lagrangian submanifold of a complex space form. We define the *Maslov form* Φ of M as

$$\Phi_p : T_pM \to \mathbb{R} : X \mapsto \langle X, JH \rangle,$$

i.e. the dual form of JH.

Proposition 8.1.2. *Let M be a Lagrangian submanifold of a complex space form $\tilde{M}(4\tilde{c})$. Then the Maslov form Φ on M is closed, i.e. $d\Phi \equiv 0$ [Che98].*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame for TM, and extend it with $\{e_{n+1} = Je_1, \dots, e_{2n} = Je_n\}$ to form an orthogonal frame for $T\tilde{M}$. Let ω^i be the dual forms to $\{e_1, \dots, e_n\}$ and $\tilde{\omega}_i^j$ the connection forms of $\tilde{\nabla}$.

Choosing $j = i$ in (2.3.3) gives us

$$d\tilde{\omega}_i^{i+n} = 2\sum_k \tilde{\omega}_k^i \wedge \tilde{\omega}_{i+n}^k + \tilde{\Omega}_i^{i+n}. \tag{8.1.1}$$

Now consider the Maslov form Φ:

$$\begin{aligned}\Phi = \langle ., JH \rangle &= \sum_j \langle e_j, JH \rangle \omega^j = -\frac{1}{n}\sum_{i,j} \langle h(e_i, e_i), Je_j \rangle \omega^j = -\frac{1}{n}\sum_{i,j} \langle h(e_j, e_i), Je_i \rangle \omega^j \\ &= -\frac{1}{n}\sum_{i,j} \langle \tilde{\nabla}_{e_j} e_i, Je_i \rangle \omega^j = -\frac{1}{n}\sum_{i,j} \omega_i^{i+n}(e_j)\omega^j = -\frac{1}{n}\sum_i \tilde{\omega}_i^{i+n}. \end{aligned}\tag{8.1.2}$$

Therefore, combining (8.1.1) and (8.1.2) we find that

$$d\Phi = -\frac{1}{n}\sum_i d\tilde{\omega}_i^{i+n} = -\frac{1}{n}\left(2\sum_{i,k} \tilde{\omega}_k^i \wedge \tilde{\omega}_{i+n}^k + \sum_i \tilde{\Omega}_i^{i+n}\right).$$

Now, we show that the first term vanishes:

$$\begin{aligned}\sum_{i,k}\tilde{\omega}_k^i\wedge\tilde{\omega}_{i+n}^k &= \sum_{i\neq k}\tilde{\omega}_k^i\wedge\tilde{\omega}_{i+n}^k = \sum_{i<k}\tilde{\omega}_k^i\wedge\tilde{\omega}_{i+n}^k + \sum_{k<i}\tilde{\omega}_k^i\wedge\tilde{\omega}_{i+n}^k\\ &= -\sum_{i<k}\tilde{\omega}_i^k\wedge\tilde{\omega}_{k+n}^i + \sum_{i<k}\tilde{\omega}_i^k\wedge\tilde{\omega}_{k+n}^i = 0.\end{aligned}$$

And so does the second term:

$$\begin{aligned}\sum_i\tilde{\Omega}_i^{i+n} &= \frac{1}{2}\sum_{i,j,k}\langle\tilde{R}(e_{i+n},e_i)e_j,e_k\rangle\omega^j\wedge\omega^k = \frac{1}{2}\sum_{i,j,k}\langle\tilde{R}(Je_i,e_i)e_j,e_k\rangle\omega^j\wedge\omega^k\\ &= -\frac{1}{2}\sum_{i,j,k}\Big(\langle\tilde{R}(e_i,e_j)Je_i,e_k\rangle + \langle\tilde{R}(e_j,Je_i)e_i,e_k\rangle\Big)\omega^j\wedge\omega^k\\ &= -\frac{1}{2}\sum_{i,j,k}\Big(\langle\tilde{R}(e_i,e_j)Je_k,e_i\rangle + \langle\tilde{R}(Je_i,e_j)Je_k,Je_i\rangle\Big)\omega^j\wedge\omega^k\\ &= -\frac{1}{2}\sum_{j,k}\sum_{\alpha}\langle\tilde{R}(e_\alpha,e_j)Je_k,e_\alpha\rangle\omega^j\wedge\omega^k = -\frac{1}{2}\sum_{j,k}\widetilde{\mathrm{Ric}}(e_j,Je_k)\omega^j\wedge\omega^k = 0.\end{aligned}$$

Thus $d\Phi\equiv 0$ and therefore the Maslov form Φ is closed. □

Theorem 8.1.3. *Let M^n be a compact Lagrangian submanifold of a complex space form $\tilde{M}^n(4\tilde{c})$. If M has finite fundamental group $\pi_1(M)$ or null first Betti number $b_1(M)$, then M has minimal points, i.e. the mean curvature H must vanish at some points.*

Proof. Assume that H is nowhere zero, then the Maslov form Φ is nowhere zero. Since Φ is a closed 1-form, its equivalence class $[\Phi]\in H^1_{dR}(M)$. Now suppose that Φ is exact, then there exists a function $f: M\to\mathbb{R}$ such that $\Phi = df$. But M is compact, so f attains a maximum on M, say at p. Then $\Phi_p = (df)_p = 0$, which is a contradiction with Φ being nowhere zero. So Φ is not exact, so its equivalence class is nontrivial and thus $H^1(M,\mathbb{R})$ is nontrivial. This means that $b_1(M) = \dim H^1_{dR}(M) > 0$ and thus M is not simply connected. We now consider 2 cases:

(i) $\pi_1(M) = 0$. We know that $b_1(M)\neq 0$, thus M is not simply connected. But then $\pi_1(M)\neq 0$, which is a contradiction. So M has a minimal point p,

(ii) $\pi_1(M)\neq 0$. Let $\phi: M\to\tilde{M}$ be the Lagrangian immersion of M into $\tilde{M}$. Denote by $\hat{M}$ the universal Riemannian covering space of M and let $\pi:\hat{M}\to M$ be the universal covering map. Then N is also a compact n-dimensional manifold since M is compact and $\pi_1(M)<\infty$. Then $\hat{\phi} = \phi\circ\pi$ is a Lagrangian immersion of $\hat{M}$ into $\tilde{M}$. But $\hat{M}$ is simply connected, thus $\pi_1(\hat{M}) = 0$. But by case (i) we then find that $\hat{M}$ has a minimal point $\hat{p}$. Then $p = \pi(\hat{p})$ is a minimal point of M.

Thus M has at least one point p such that $H(p) = 0$. □

Theorem 8.1.3 is sharp:

Example 8.1.4. Consider the standard embedding

$$\phi : T^n = S^1 \times \cdots \times S^1 \to \mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}.$$

T^n is a compact Lagrangian submanifold with nonzero constant mean curvature. However, T^n has first Betti number $b_1(T^n) = n$, and fundamental group $\pi_1(T^n) = \mathbb{Z}^n$ which is infinite.

Corollary 8.1.5. *There do not exist Lagrangian immersions of nonzero constant mean curvature n-spheres into complex space forms.*

Proof. Follows directly from theorem 8.1.3. □

Before stating the next corollary, we need the following result:

Lemma 8.1.6. *A complete n-dimensional Riemannian manifold M for which* $\mathrm{Ric}(X) \geq c > 0$ *for any $X \in UM$, is compact and has finite fundamental group [Mye41].*

Corollary 8.1.7. *If M is a complete Riemannian manifold whose Ricci curvature satisfies* $\mathrm{Ric}(X) \geq c > 0$ *for any $X \in UM$, then every Lagrangian immersion of constant mean curvature of M into a complex space form $\tilde{M}$ is a minimal immersion.*

Proof. By lemma 8.1.6, we know M is compact and has finite fundamental group. Thus we can apply theorem 8.1.3 to find that there exists a minimal point. But since H is constant, we then know that M is minimal. □

Corollary 8.1.8. *Let $M^n(c)$ be a compact real space form of positive constant curvature $c > 0$, immersed as a Lagrangian submanifold with constant mean curvature in $\mathbb{C}P^n$. Then M is totally geodesic.*

Proof. Since M is a compact real space form of positive curvature, it has finite fundamental group. By theorem 8.1.3 M is a minimal immersion, and by theorem 3.2.8 it is totally geodesic. □

Corollary 8.1.9. *There do not exist Lagrangian isometric immersions from a compact Riemannian manifold with positive Ricci curvature into $\mathbb{C}^n$ or $\mathbb{C}H^n$.*

Proof. Let M be a compact Riemannian manifold with positive Ricci curvature. If $\phi : M \to \tilde{M}$ is a Lagrangian isometric immersion of M into a complex space form $\tilde{M}(4\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c} \leq 0$, then by theorem 8.1.3 ϕ has at least one minimal point, say at $p \in M$.

Now, consider the equation of Gauss (4.2.2), which gives that

$$\tilde{c}\langle (e_i \wedge e_j)e_j, e_i\rangle = \langle R(e_i, X)Y, e_i\rangle - \langle h(e_i, e_i), h(e_j, e_j)\rangle + \langle h(e_i, e_j), h(e_i, e_j)\rangle.$$

Summing over all $i = 1, \ldots, n$ gives

$$\tilde{c}(n-1) = \mathrm{Ric}(e_j) - n\langle H, h(e_j, e_j)\rangle + \sum_{i=1}^{n} \|h(e_i, e_j)\|^2.$$

Let us assume $H = 0$ at some point p, then we can rearrange this equation at p as

$$\mathrm{Ric}(e_j) = \tilde{c}(n-1) - \sum_{i=1}^{n} \|h(e_i, e_j)\|^2.$$

However, since $\tilde{c} \leq 0$, the right-hand side is negative whereas the left-hand side is strictly positive, which is a contradiction. □

We now show how we can use δ-invariants in this situation:

Corollary 8.1.10. *Let M^n be a compact Riemannian manifold with first Betti number $b_1(M) = 0$ or with finite fundamental group $\pi_1(M)$, and let $\tilde{c} \in \mathbb{R}$. If there exists a k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ such that $\delta(n_1, \ldots, n_k) > b(n_1, \ldots, n_k)\tilde{c}$ at every point of M, then M does not admit any Lagrangian isometric immersion into a complex space form $\tilde{M}(4\tilde{c})$.*

Proof. Assume that M admits a Lagrangian isometric immersion $\phi : M \to \tilde{M}^n(4\tilde{c})$. If M satisfies $\delta(n_1, \ldots, n_k) > b(n_1, \ldots, n_k)\tilde{c}$ for some k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, then by theorem 6.3.2 we find

$$b(n_1, \ldots, n_k)\tilde{c} < \delta(n_1, \ldots, n_k) \leq c(n_1, \ldots, n_k)\|H\|^2 + b(n_1, \ldots, n_k)\tilde{c},$$

and thus

$$0 < c(n_1, \ldots, n_k)\|H\|^2.$$

Since $c(n_1, \ldots, n_k)$ is strictly positive, we find that H is nowhere zero. But then theorem 8.1.3 gives a contradiction, thus such immersion does not exist. □

Remark 8.1.11. The condition on $\delta(n_1, \ldots, n_k)$ in corollary 8.1.10 is sharp. For example, consider the Whitney sphere in example 3.3.11. This is a Lagrangian isometric immersion with $\delta(n_1, \ldots, n_k) \geq 0$ at all points, with equality holding only at the unique point of self-intersection $w(1, 0, \ldots, 0) = w(-1, 0, \ldots, 0)$.

The assumptions on $\pi_1(M)$ and $b_1(M)$ in corollary 8.1.10 are both necessary if $n \geq 3$:

Example 8.1.12. Consider the unit circle $F : S^1 \to \mathbb{C} : s \mapsto e^{is}$, and let $\iota : S^{n-1} \to \mathbb{E}^n$ $(n \geq 3)$ be the unit hypersphere in $\mathbb{E}^n$ centred at the origin. Then

$$\phi : M = S^1 \times S^{n-1} \to \mathbb{C}^n : (s, p) \mapsto F(s) \otimes \iota(p),$$

is a complex extensor, thus a Lagrangian isometric immersion of M into $\mathbb{C}^n$ which carries each pair of points $\{(s, p), (-s, -p)\} \in M$ to a point in $\mathbb{C}^n$. Clearly $\pi(M) = \mathbb{Z}$ and $b_1(M) = 1$. Moreover, for each k-tuple $(n_1, \ldots, n_k)$, the δ-invariant $\delta(n_1, \ldots, n_k)$ is a positive constant.

8.2 Results for $k = 0$ or $k = 1$

We first consider the case $k = 0$. We can provide an improvement of corollary 6.4.1 using the improved inequality.

Corollary 8.2.1. *Let M^n be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4\tilde{c})$. Then the scalar curvature of a M satisfies*

$$\tau \leq \frac{n(n-1)}{2}\left(\frac{n}{n+2}\|H\|^2 + \tilde{c}\right). \tag{8.2.1}$$

The equality sign of (8.2.1) *holds if and only if M is a Lagrangian H-umbilical submanifold with $\lambda = 3\mu$.*

Proof. If we choose $k = 0$ (and therefore $n = n_{k+1}$), then (7.2.3) reduces immediately to (8.2.1). For the equality case of the theorem, note that $T_pM = L_{k+1}$. We know that $C_{rrr} = 3C_{rss}$ for $r \neq s$ and $C_{rst} = 0$ for r, s, t all different.

Now first suppose that M is minimal. Then for any r, t,

$$0 = \langle H, Je_r\rangle = \sum_{s=1}^{n} C_{rss} = \left(1 + \frac{n-1}{3}\right) C_{rrr} = (n+2)C_{rtt},$$

thus $C \equiv 0$ and M is totally geodesic. If M is not minimal, we can choose $e_1 = -JH/\|H\|$. So for any $r \neq 1$:

$$\|H\| = \langle H, Je_1\rangle = \sum_{s=1}^{n} C_{1ss} = \left(1 + \frac{n-1}{3}\right) C_{111} = (n+2)C_{1rr},$$

and for any r, t different and not 1:

$$0 = \langle H, Je_r\rangle = \sum_{s=1}^{n} C_{rss} = \left(1 + \frac{n-1}{3}\right) C_{rrr} = (n+2)C_{rtt},$$

so M is H-umbilical. □

Remark 8.2.2. Inequality (8.2.1) was already known before the improved inequality for Lagrangian submanifolds. For $n = 2$ it was proved in [CU93] ($\tilde{c} = 0$) and [CU95] ($\tilde{c} \neq 0$). For general n it was proved in [BCM95] ($\tilde{c} = 0$) and [Che96; CV96] ($\tilde{c} \neq 0$).

Next, let us give a general result for $k = 1$:

Theorem 8.2.3. *Let M be a Lagrangian submanifold of a complex space form $M^n(4\tilde{c})$. Then for any integer $n_1 \in \{2, \ldots, n-1\}$ we have*

$$\delta(n_1) \leq \frac{n^2(n_1(n-n_1) + 2(n-1))}{2(n_1(n-n_1) + 2n + 3n_1 + 4)}\|H\|^2 + \frac{1}{2}(n(n-1) - n_1(n_1-1))\tilde{c}. \tag{8.2.2}$$

Moreover, if the equality sign of (8.2.2) *holds identically for some $n_1 \leq n-2$, then M is a minimal submanifold [CD11b].*

Proof. The inequality follows immediately from taking $k = 1$ in (7.2.3). For the equality case, we refer to [CD11b]. □

Remark 8.2.4. Theorem 8.2.3 is sharp. Let $F : I \to \mathbb{C}^* : s \mapsto r(s)e^{i\theta(s)}$ be a unit speed curve with curvature $\kappa(s) = (n+1)\theta'(s) \neq 0$ and let $\iota : S^{n-1}(1) \to \mathbb{E}^n$ be the unit hypersphere centred at the origin. Then the complex extensor $F \otimes \iota : I \times S^{n-1}(1) \to \mathbb{C}^n$ is a non-minimal Lagrangian submanifold satisfying equality in (8.2.2) [CD11b]. An example of a non-minimal Lagrangian submanifold of $\mathbb{C}P^3(4\tilde{c})$ attaining equality can be found in [BV07].

For the Ricci curvature we find the following improvement of corollary 6.4.2:

Proposition 8.2.5. *Let M^n be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4\tilde{c})$. Then the Ricci curvature of M satisfies*

$$\mathrm{Ric}(X) \leq \frac{n(n-1)}{4}\|H\|^2 + (n-1)\tilde{c}. \tag{8.2.3}$$

For any $X \in UM$. Equality holds for every X if and only if M is either totally geodesic or a H-umbilical submanifold with $\lambda = 3\mu$ [Den09; Opr05].

Proof. Let $n_1 = n - 1$ in (8.2.2). Then we obtain

$$\mathrm{Ric}(X) \leq \max_{\|Y\|=1} \mathrm{Ric}(Y) = \delta(n-1)\frac{n(n-1)}{4}\|H\|^2 + (n-1)\tilde{c}.$$

For the equality case, we refer to [Den09]. □

To finish, we mention that several results have been obtained for the $\delta(2)$-invariant, for example see [BDV07; BV07; Che+94; Che+96; CV02; Dil+14; Opr08].

Appendix A

Lagrangian submanifolds in symplectic geometry

In this chapter, we give a small introduction to symplectic geometry and in particular we discuss Lagrangian submanifolds from the symplectic point of view. This chapter is based on [Arn89; Arn90; DS01; Wei77].

Definition A.1. Let M be a differentiable manifold, and let $\omega \in \Omega^2(M)$ with the following properties:

(i) ω is nondegenerate (or symplectic), i.e. if $\omega(X, Y) = 0\ \forall Y$, then $X = 0$,

(ii) ω is closed, i.e. $d\omega = 0$,

then we call (M, ω) a *symplectic manifold* and ω is called the *symplectic form*.

Note that a necessary condition for ω to be symplectic is that $\dim_{\mathbb{R}} M$ is even.

Suppose M has an almost complex structure J. Then we say that J is *compatible* with the symplectic form ω if $\langle ., . \rangle = \omega(., J.)$ is a Riemannian metric on M.

In symplectic geometry, the definition of a Kähler manifold is as follows:

Definition A.2. A *Kähler manifold* is a symplectic manifold (M, ω) equipped with a compatible almost complex structure J such that $\mathcal{N}_J \equiv 0$. The symplectic form ω is then called a *Kähler form*.

Remark A.3. From the closedness of ω follows that $\nabla J \equiv 0$, so M is clearly a Kähler manifold in the Riemannian sense. But the converse holds too: if we have a Kähler manifold M in the Riemannian sense and we define $\omega(., .) = \langle J., . \rangle$, then from $\nabla J \equiv 0$ follows that ω is closed and thus M is Kähler in the symplectic sense. So both definitions are equivalent.

A Lagrangian submanifold is defined as follows:

Definition A.4. Let M be a submanifold that has half the dimension of the ambient symplectic manifold $(\tilde{M}, \omega)$, such that $\omega|_M \equiv 0$. Then we call M a *Lagrangian submanifold*.

Remark A.5. Since $\omega(.,.) = \langle J.,.\rangle$, this means that $\langle JX, Y\rangle = 0 = \omega(X, Y)$ for $X, Y \in T_pM$. So again this definition is equivalent to the Riemannian definition.

So where does the big difference between Riemannian and symplectic geometry lie? In Riemannian geometry, we defined (sub)manifolds up to isometry. However, in symplectic geometry, we consider things up to symplectomorphism:

Definition A.6. Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds and let $\phi : M_1 \to M_2$ be a diffeomorphism. Then ϕ is a *symplectomorphism* if $\phi^*\omega_2 = \omega_1$.

We have the following theorem:

Theorem A.7 (Darboux)**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold, and let p be any point in M. Then there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centred at p such that on $\mathcal{U}$*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

If we have 2 Lagrangian submanifolds M_1^n and M_2^n of a symplectic manifold $(\tilde{M}^n, \omega)$ and points p and q on them respectively, there is a symplectomorphism from a neighbourhood of p (in the ambient symplectic manifold $\tilde{M}$) to a neighbourhood of q which sends p to q and maps M_1 to M_2. This can be done using Darboux's theorem: simply take a coordinate basis such that the first n coordinates vanish on the Lagrangian submanifold.

Conclusions and further research

In this thesis we have given the reader an introduction to Lagrangian submanifolds of complex space forms. In the preliminaries, we gave a description of complex space forms and Lagrangian submanifolds, and we discussed their basic properties. A canonical basis of the tangent space was introduced, which has several nice properties helpful in determining the second fundamental form, and thus the properties of the submanifold itself.

In the first part of this thesis we discussed parallelity conditions we may impose on a Lagrangian submanifold. Basic properties of each of these conditions were considered, as well as how the various conditions interact with each other. In particular, we researched the condition of pseudo-parallel cubic form proposed in [DVV09] and introduced the new condition of H-pseudo-parallelity. We also discussed the use of the canonical basis to decompose the tangent space, and gave some classification results.

In the second part of this thesis we considered Chen's δ-invariants and the inequality we can obtain between these invariants and the mean curvature. We then restricted ourselves to the Lagrangian case, providing a better inequality and giving corollaries of this improved inequality.

The work in this thesis can be continued and extended in many ways. We give a short list of suggestions:

(i) For Lagrangian submanifolds, we introduced a normal wedge $\wedge^{\perp}$ operator and a Lagrangian wedge $\bar{\wedge} = \wedge \oplus \wedge^{\perp}$. The reason to do so was that this construction retains the same symmetries as the Van der Waerden-Bortolotti connection $\bar{\nabla}$ and curvature $\bar{R}$. However, the construction of the normal wedge was strongly based on the complex structure J. One may wonder if it possible to define similar operators in different or more general settings.

(ii) In the case of Lagrangian surfaces, we have shown that the 'weakest' conditions were constant curvature K and H-umbilicity, both of which have been classified. Is it possible to classify all Lagrangian surfaces, or can we come up with a weaker condition than the two aforementioned ones to classify?

(iii) Can the given technique for decomposing the tangent space be improved upon, and can it be applied to give more classifications of certain classes of Lagrangian submanifolds?

(iv) Finally, the improved inequality for Lagrangian submanifolds of complex space forms was only proven very recently, in 2013. There is still much work to be done for the classification of (improved) ideal Lagrangian submanifolds.

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