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Reidemeister spectra for almost-crystallographic groups

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Dissertation presented in partial fulfilment of the requirements for the degree of Doctor of Science (PhD): Mathematics

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Abstract

The notion of conjugacy in a group can be generalised to *twisted conjugacy.* For any endomorphism φ of a group G, we may define an equivalence relation $\sim_\varphi$ on G by

$$\forall g, g' \in G : g \sim_\varphi g' \iff \exists h \in G : g = hg'\varphi(h)^{-1}.$$

The number of equivalence classes is called the *Reidemeister number* and is denoted by $R(\varphi)$. The set of all possible Reidemeister numbers of automorphisms is called the *Reidemeister spectrum.*

This notion originates in topological fixed-point theory. A continuous self-map f on a (sufficiently well-behaved) topological space X induces an endomorphism f_* on the fundamental group $\pi_1(X)$. The Reidemeister number $R(f_*)$ is an upper bound for the Nielsen number $N(f)$, which in turn is a lower bound for the number of fixed points of f.

In this thesis, we investigate the Reidemeister spectra of almost-crystallographic groups. These groups are generalisations of the crystallographic groups, in the sense that their translation subgroup is nilpotent rather than abelian. The main results can be grouped into two parts.

In the first part, we investigate the Reidemeister spectra of finitely generated, torsion-free, nilpotent groups. We compute the spectrum for every such group of dimension at most 4. Furthermore, we compute the Reidemeister spectra of free nilpotent groups of low rank and/or nilpotency class.

In the second part, we first determine which low-dimensional almost-crystallographic groups admit automorphisms with finite Reidemeister number. Next, we provide an algorithm that is capable of calculating the Reidemeister number of any given automorphism of a crystallographic group, and use this to calculate the Reidemeister spectra. Finally, we determine which almost-crystallographic groups admit Reidemeister zeta functions, and prove that these functions are rational for groups of dimension at most 3.

Beknopte samenvatting

De notie van conjugatie in een groep kan worden veralgemeend naar *getwiste conjugatie*. Voor elk endomorfisme φ van een groep G, kunnen we een equivalentierelatie $\sim_\varphi$ op G definiëren als

$$\forall g, g' \in G : g \sim_\varphi g' \iff \exists h \in G : g = hg'\varphi(h)^{-1}.$$

Het aantal equivalentieklassen wordt het *Reidemeistergetal* genoemd en wordt genoteerd met $R(\varphi)$. De verzameling van alle mogelijke Reidemeistergetallen van automorfismen wordt het *Reidemeisterspectrum* genoemd.

Deze notie vindt zijn oorsprong in de topologische vastepuntstheorie. Een continue zelf-afbeelding f op een (voldoende brave) topologische ruimte X induceert een endomorfisme f_* op de fundamentaalgroep $\pi_1(X)$. Het Reidemeistergetal $R(f_*)$ is een bovengrens voor het Nielsengetal $N(f)$, dat op zijn beurt een ondergrens is voor het aantal vaste punten van f.

In deze thesis onderzoeken we de Reidemeisterspectra van bijna-kristallografische groepen. Deze groepen zijn veralgemeningen van de kristallografische groepen, in die zin dat hun translatiedeelgroep nilpotent is in plaats van abels. De belangrijkste resultaten kunnen in twee delen worden gegroepeerd.

In het eerste deel onderzoeken we de Reidemeisterspectra van eindig voortgebrachte, torsievrije, nilpotente groepen. We berekenen het spectrum voor deze groepen met dimensie maximaal 4. Verder berekenen we de Reidemeisterspectra van vrije nilpotente groepen van lage rang en/of nilpotentieklasse.

In het tweede deel bepalen we eerst welke laag-dimensionale bijna-kristallografische groepen automorfismen met eindig Reidemeistergetal toelaten. Vervolgens geven we een algoritme dat in staat is om het Reidemeistergetal van een gegeven automorfisme van een kristallografische groep te berekenen en gebruiken dit om de Reidemeisterspectra te berekenen. Ten slotte bepalen we welke bijna-kristallografische groepen Reidemeister-zèta-functies toelaten, en bewijzen we dat deze functies rationaal zijn voor groepen met dimensie maximaal 3.

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Chapter 1

Introduction

The main goal of fixed point theory is, given a self-map $f : X \to X$ on a space X, to find the fixed point set

$$\mathrm{Fix}(f) := \{x \in X \mid f(x) = x\}.$$

Fixed point theory is a branch of mathematics that intersects with many other mathematical domains. For example, we have the Brouwer fixed point theorem in topology and the Banach fixed point theorem in analysis.

Fixed point theory also yields a multitude of applications in a wide variety of other scientific domains. If one describes an iterative process using a function, then the equilibria of this process will coincide with the fixed points of the function. Thus, finding the fixed points will provide information on the asymptotic behaviour of the iterative process.

One example is the predator-prey model in biology. This model describes a system in which two species interact, one as predator and one as prey. If the evolution of their populations are described by an iterative process, then the fixed points will indicate that either the populations remain stable or that they go extinct. Another example is that of the Nash equilibrium in non-cooperative games, whose existence was proven by John Nash using the Brouwer fixed point theorem. His work on this equilibrium won Nash the Nobel Memorial Prize in Economic Sciences.

1.1 Reidemeister-Nielsen fixed point theory

Fixed points are often quite hard to find, if they exist at all. Moreover, a slight modification to the function f may completely change the set of fixed points. Topological fixed point theory attempts to resolve these problems by asking the following questions:

- Does every map homotopic to f have at least one fixed point?
- How many fixed points must every map homotopic to f at least have?

In the 1880's, Henri Poincaré was the first to introduce topological methods in the study of non-linear analysis, and in particular the topological study of fixed points. This sparked the discovery of several fixed point theorems. While studying fixed points in the 1900's and early 1910's, Luitzen Egbertus Jan Brouwer proved the *Brouwer fixed point theorem*, which answers the first question for closed disks.

Theorem. *A continuous self-map on a closed disk has at least one fixed point.*

In the early 1920's, Solomon Lefschetz generalised this result by assigning a homotopy-invariant integer $L(f)$ (now called the Lefschetz number) to a continuous self-map f on a compact, connected polyhedron X. He defined this number as

$$L(f) := \sum_{k=0}^{\dim X} (-1)^k \operatorname{tr}\left(f_{k,*} : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})\right).$$

While the Lefschetz number does not coincide with the number of fixed points of f, it does give information on the existence of a fixed point by means of the *Lefschetz fixed point theorem.*

Theorem. *Let f be a continuous self-map on a compact, connected polyhedron. If $L(f) \neq 0$, then (any map homotopic to) f has at least one fixed point.*

Still in the 1920's, Jakob Nielsen worked on answering the second question. In contrast to the algebraic approach of Lefschetz, he devised a geometric way to count the fixed points. His approach can be summarised in three steps:

1. Partition $\operatorname{Fix}(f)$ into fixed point classes.
2. Determine which fixed point classes are essential, i.e. they cannot vanish under homotopies.

3. Count the number of essential fixed point classes.

The number of essential fixed point classes of f is now called the Nielsen number $N(f)$. By definition, an essential fixed point class contains at least one fixed point, hence the Nielsen number is a homotopy-invariant lower bound for the number of fixed points of f.

Of course, we have not yet mentioned how we partition $\mathrm{Fix}(f)$ and how we determine whether or not a fixed point class is essential. Nielsen noted that, when considering lifts of f to the universal cover $\tilde{X}$ of X, that these lifts behave very different in general, yet very similar when they are conjugate by an element of the covering transformation group $\mathcal{D}(X)$. He therefore introduced the equivalence relation on the set of lifts of f given by

$$\tilde{f}_1 \sim \tilde{f}_2 \iff \exists \gamma \in \mathcal{D}(X) : \tilde{f}_1 = \gamma \circ \tilde{f}_2 \circ \gamma^{-1}.$$

The fixed points of the lifts $\tilde{f}$ completely determine the fixed point set $\mathrm{Fix}(f)$, in a way that behaves nicely under the above equivalence:

$$\mathrm{Fix}(f) = \bigsqcup_{[\tilde{f}]} p(\mathrm{Fix}(\tilde{f})),$$

where $[\tilde{f}]$ is the equivalence class containing the lift $\tilde{f}$ and $p : \tilde{X} \to X$ is the covering map from $\tilde{X}$ to X. The fixed point classes are exactly the sets $p(\mathrm{Fix}(\tilde{f}))$.

Nielsen then assigned an integer to each fixed point class, called the fixed point index, such that the fixed point class is essential if and only if the fixed point index is non-zero. This index is closely related to the Lefschetz number, as illustrated by the *Lefschetz-Hopf fixed point theorem*, proven by Heinz Hopf in the late 1920's.

Theorem. *Let f be a continuous self-map on a compact, connected polyhedron. Then the Lefschetz number $L(f)$ is the sum of the fixed point indices of the fixed point classes of f.*

In the 1930's and 1940's, Kurt Reidemeister and his student Franz Wecken again took a more algebraic approach to studying fixed points. They noted that, since any lift $\tilde{f}$ induces an endomorphism $f_* : \mathcal{D}(X) \to \mathcal{D}(X)$ by

$$\tilde{f} \circ \gamma = f_*(\gamma) \circ \tilde{f} \quad \forall \gamma \in \mathcal{D}(X),$$

the above equivalence relation induces an equivalence relation called f_*-twisted conjugacy on $\mathcal{D}(X)$:

$$\alpha_1 \sim \alpha_2 \iff \exists \gamma \in \mathcal{D}(X) : \alpha_1 = \gamma \alpha_2 f_*(\gamma)^{-1}.$$

Thus, the number of fixed point classes of f is the same as the number of f_*-twisted conjugacy classes of $\mathcal{D}(X)$, which is called the Reidemeister number $R(f)$. Since the Reidemeister number counts both the essential and inessential fixed point classes, it is an upper bound for the Nielsen number:

$$N(f) \leq R(f).$$

Twisted conjugacy can be defined for any group G and any endomorphism $\varphi : G \to G$: we define the equivalence relation $\sim_\varphi$ by

$$\forall g, g' \in G : g \sim_\varphi g' \iff \exists h \in G : g = hg'\varphi(h)^{-1}.$$

The number of equivalence classes is again called the Reidemeister number and denoted by $R(\varphi)$. The Reidemeister spectrum of a group is the set

$$\mathrm{Spec}_R(G) = \{R(\varphi) \mid \varphi \in \mathrm{Aut}(G)\},$$

and we say that G has the R_∞-property if $\mathrm{Spec}_R(G) = \{\infty\}$.

In the 1960's, Stephen Smale introduced the Lefschetz zeta function $L_f(z)$ of a self-map f, defined as

$$L_f(z) := \exp \sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n,$$

and proved that this function is rational. In the 1990's, Alexander Fel'shtyn defined the Nielsen and Reidemeister zeta functions of a self-map f analogously as

$$N_f(z) := \exp \sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n,$$

$$R_f(z) := \exp \sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n.$$

Unlike the Lefschetz zeta function, the Nielsen and Reidemeister zeta functions need not be rational in general. Since the Reidemeister number can be defined for any group G and endomorphism φ, we can also define the Reidemeister zeta function of φ:

$$R_\varphi(z) := \exp \sum_{n=1}^{\infty} \frac{R(\varphi^n)}{n} z^n.$$

Central to this thesis are the R_∞-property, the Reidemeister spectrum and the Reidemeister zeta functions.

1.2 Overview of the thesis and main results

This thesis consists of three main parts and two appendices.

Part I. The first part forms an introduction, where we review the topics needed to understand this thesis. In chapter 2, we give a brief overview of Reidemeister-Nielsen fixed point theory. We mainly focus on the Reidemeister number, both in its topological and group-theoretical setting. In chapter 3 we define nilpotent groups, Lie groups (and algebras), crystallographic groups and almost-crystallographic groups. Finally, in chapter 4, we mention the main results in Reidemeister-Nielsen fixed point theory on almost-crystallographic groups.

Part II. The second part of this thesis focuses on the R_∞-property and Reidemeister spectrum of nilpotent groups. Chapter 5 deals with the finitely generated, torsion-free, nilpotent groups of dimension at most 4. For each of these groups, we completely determine the Reidemeister spectrum. Chapter 6 deals with a particular subset of the finitely generated, torsion-free, nilpotent groups, namely the free nilpotent groups. We determine the Reidemeister spectra of the free nilpotent groups of nilpotency class 2, obtaining the following result:

Theorem. *A free nilpotent group of rank at least* 4 *and nilpotency class* 2 *has full Reidemeister spectrum.*

We also determine the Reidemeister spectra of the free nilpotent groups of rank 2 and 3. At the end of this chapter, we consider direct products of free nilpotent groups, proving that the Reidemeister spectrum of such product is determined completely by the spectra of its factors.

Part III. The final part of this thesis, which is by far the lengthiest part, focuses on the R_∞-property, Reidemeister spectra and Reidemeister zeta functions of (low-dimensional) almost-crystallographic groups.

Chapters 7 and 8 each study a specific family of crystallographic groups, namely the crystallographic groups with diagonal holonomy $\mathbb{Z}_2$ and the generalised Hantzsche-Wendt groups respectively. For both families, we find necessary and sufficient conditions for a group to have the R_∞-property, we calculate the Reidemeister spectra, determine when they admit Reidemeister zeta functions and prove the rationality of these zeta functions.

In chapter 9 we study the R_∞-property for low-dimensional almost-crystallographic groups. For the crystallographic groups with finite outer automorphism group, we provide an algorithm that determines whether the group has the R_∞-property; for the crystallographic groups with infinite outer automorphism group and the non-crystallographic almost-crystallographic groups we use ad hoc methods. For the following groups, we determine whether or not they have the R_∞-property:

- the almost-crystallographic groups of dimension at most 4,
- the crystallographic groups of dimension at most 6 whose outer automorphism group is finite.

In chapter 10 we study the Reidemeister spectra of the low-dimensional almost-crystallographic groups that do not have the R_∞-property, as determined in the previous chapter. For the crystallographic groups with finite outer automorphism group, we provide an algorithm that calculates the Reidemeister spectrum; for the crystallographic groups with infinite outer automorphism group and the non-crystallographic almost-crystallographic groups we again use ad hoc methods. We calculate the Reidemeister spectra of the following groups:

- the almost-crystallographic groups of dimension at most 3,
- the almost-Bieberbach groups of dimension at most 4,
- the crystallographic groups of dimension at most 6 whose outer automorphism group is finite.

In chapter 11 we study the existence and rationality of Reidemeister zeta functions of the low-dimensional almost-crystallographic groups. We determine which almost-crystallographic groups of dimension at most 3 admit Reidemeister zeta functions, and obtain the following result regarding their rationality.

Theorem. *A Reidemeister zeta function of an almost-crystallographic group of dimension at most* 3 *is rational.*

Appendix. There are two appendices.

Appendix A is about isogredience, a concept closely related to twisted conjugacy. For the following groups, we determine whether or not they have the S_∞-property:

- the almost-crystallographic groups of dimension at most 3,

- the crystallographic groups of dimension at most 4.

We also calculate the isogredience spectra of the almost-crystallographic groups of dimension at most 3.

Appendix B simply contains tables that are referenced throughout this thesis, but whose inclusion in the relevant chapters would have seriously hampered their readability.

Part I

Preliminaries

Chapter 2

Reidemeister-Nielsen fixed point theory

Given a function $f : X \to X$ on some topological space X, the goal of topological fixed point theory is to answer the following question: "Does every map g homotopic to f admit fixed points, and if so, how many fixed points must g have at the very least?". If the space X is sufficiently nice (e.g. a compact polyhedron or manifold), we can define the integers $L(f)$, $R(f)$ and $N(f)$, each of which provides certain information pertinent to this question.

The goal of this chapter is to give a quick overview of the ideas and results in Reidemeister-Nielsen fixed point theory. Since the focus of this thesis is on Reidemeister numbers, we will mostly provide proofs of theorems when they are relevant to the Reidemeister number, and omit them otherwise. The interested reader can find more complete and more detailed expositions in [Bro+05; Jia83; Tsa89].

2.1 The Lefschetz number

One of the first tools to study the existence of a fixed point is the Lefschetz number. This concept was introduced by Solomon Lefschetz in a series of papers [Lef23; Lef25; Lef26; Lef27].

Definition 2.1.1. Let $f : X \to X$ be a continuous self-map on a closed, connected polyhedron X. The *Lefschetz number* $L(f)$ is defined as

$$L(f) := \sum_{k=0}^{\dim X} (-1)^k \operatorname{tr}\left(f_{k,*} : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})\right),$$

where $f_{k,*}$ is the induced morphism on the k-th homology group $H_k(X, \mathbb{Q})$.

Note that the Lefschetz number is a generalisation of the Euler characteristic, since $L(\mathrm{id}_X) = \chi(X)$. Moreover, it is invariant under homotopy, since homotopic maps will induce the same morphisms on the homology groups. As we already mentioned, the Lefschetz number allows us to study the existence of a fixed point.

Theorem 2.1.2 (Lefschetz fixed point theorem)**.** *Let $f : X \to X$ be a continuous self-map on a connected, compact polyhedron X. If $L(f) \neq 0$, then f has at least one fixed point.*

As the Lefschetz number is homotopy invariant, each $g \simeq f$ will have at least one fixed point as well. However, the converse to the Lefschetz fixed point theorem is not necessarily true.

Example 2.1.3. The identity map id_{S^1} on the circle S^1 has Lefschetz number $L(\mathrm{id}_{S^1}) = \chi(S^1) = 0$, but obviously id_{S^1} has infinitely many fixed points.

Later, in example 4.2.2(2), we will even give an example of a continuous map f with Lefschetz number $L(f) = 0$, for which every map g homotopic to f has at least one fixed point.

2.2 Fixed point classes

Every topological space X in this section is assumed to be a connected, locally path-connected, semi-locally simply connected topological space. Such space admits a universal cover $p : \tilde{X} \to X$. Any map is also assumed to be continuous, and we denote the set of fixed points of a self-map f by $\mathrm{Fix}(f)$.

Note that throughout this thesis, we will use non-standard definitions of lifts and of induced morphisms on fundamental groups. First, let us recall the usual definition of the induced morphism.

Lemma 2.2.1 (see [Mun00, §52])**.** *Let $f : X \to Y$ be a continuous map. Let $x \in X$ and set $y = f(x)$. Then the map*

$$f_\pi : \pi_1(X, x) \mapsto \pi_1(Y, y) : [\alpha] \mapsto [f \circ \alpha]$$

is a well-defined morphism.

This map is usually denoted by f_*, however, we will use f_* for a different (but related) morphism, so we denote this one by f_π. Let us also recall the following lemma:

Lemma 2.2.2 (General lifting lemma, see [Mun00, Lemma 79.1])**.** *Let $p : Z \to Y$ be a covering map and fix $y \in Y$ and $z \in Z$ such that $p(z) = y$. Let $f : X \to Y$ be a continuous map, with $f(x) = y$. Suppose Z is path-connected and locally path connected.*

There exists a continuous map $\tilde{f} : X \to Z$ such that:

- *$p \circ \tilde{f} = f$, i.e. the diagram below commutes,*
- *$\tilde{f}(x) = z$,*

if and only if

$$f_\pi(\pi_1(X, x)) \subseteq p_\pi(\pi_1(Z, z)).$$

Furthermore, if such a map exists, it is unique.

$$\begin{array}{ccc} & & Z \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

A map $\tilde{f}$ as above is usually called a lift in the literature. However, our definition of a lift will be as follows:

Definition 2.2.3. Let $f : X \to Y$ be a continuous map between two topological spaces X, Y with universal covers $\tilde{X}$, $\tilde{Y}$ respectively. Then a *lift* $\tilde{f} : \tilde{X} \to \tilde{Y}$ of f is a map between the universal covers such that $f \circ p = p \circ \tilde{f}$. In other words, the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \big\downarrow p & & \big\downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

The existence of this lift is guaranteed by the general lifting lemma (lemma 2.2.2). Such a lift is, in general, far from unique, until we impose some extra conditions.

Lemma 2.2.4. *Let $f : X \to Y$ be a continuous map between two path-connected topological spaces X, Y admitting universal covers $\tilde{X}$, $\tilde{Y}$. Let $x \in X, y \in Y$ such that $f(x) = y$ and choose any preimages $\tilde{x} \in p^{-1}(x)$, $\tilde{y} \in p^{-1}(y)$. Then there exists a unique lift $\tilde{f}$ of f such that $\tilde{f}(\tilde{x}) = \tilde{y}$.*

Proof. Apply the general lifting lemma (lemma 2.2.2) to $f \circ p : \tilde{X} \to Y$. Since $\tilde{X}$ is simply connected, the condition for the existence of a (unique) lift is always satisfied. □

Definition 2.2.5. A lift of the identity map id_X to $\tilde{X}$ is called a *covering transformation*. The group of covering transformations will be denoted by $\mathcal{D}(X)$. Equivalently, it can be defined as the group of self-homeomorphisms γ of $\tilde{X}$ such that $p \circ \gamma = p$, with $p : \tilde{X} \to X$ the universal cover.

Some well-known properties of lifts are the following:

Proposition 2.2.6. *Let $f : X \to X$ be a self-map and $p : \tilde{X} \to X$ be the universal cover of X.*

(i) For any $x \in X$ and any $\tilde{x}, \tilde{x}' \in p^{-1}(x)$, there exists a unique covering transformation $\gamma : \tilde{X} \to \tilde{X}$ such that $\gamma(\tilde{x}) = \tilde{x}'$. In fact, $\mathcal{D}(X)$ is isomorphic to the fundamental group $\pi_1(X, x)$.

(ii) Let $x \in X$ and $x' = f(x)$. If $\tilde{x} \in p^{-1}(x)$ and $\tilde{x}' \in p^{-1}(x')$, there exists a unique lift $\tilde{f}$ such that $\tilde{f}(\tilde{x}) = \tilde{x}'$.

(iii) Let $\tilde{f}$ be a lift of f and $\alpha, \beta \in \mathcal{D}(X)$. Then $\beta \circ \tilde{f} \circ \alpha^{-1}$ is a lift of f.

(iv) Let $\tilde{f}, \tilde{f}'$ be two lifts of f. Then there is a unique $\gamma \in \mathcal{D}(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f}$.

Proof. We prove the four statements one by one.

(i) The first part of this statement is a special case of lemma 2.2.4. The second part is exactly [Mun00, Corollary 81.4]. Rather than giving the full proof, we will give an explicit isomorphism: consider the map

$$\Phi_{\tilde{x}} : \mathcal{D}(X) \to \pi_1(X, x) : \gamma \mapsto [\alpha],$$

where $\alpha = p \circ \tilde{\alpha}$ is a path in X and is the projection of a path $\tilde{\alpha}$ in $\tilde{X}$ with $\tilde{\alpha}(0) = \tilde{x}$ and $\tilde{\alpha}(1) = \gamma(\tilde{x})$.

(ii) Again, this is a special case of lemma 2.2.4.

(iii) Using that α^{-1} and β are lifts of the identity map, we obtain:

$$\begin{aligned} p \circ \beta \circ \tilde{f} \circ \alpha^{-1} &= p \circ \tilde{f} \circ \alpha^{-1} \\ &= f \circ p \circ \alpha^{-1} \\ &= f \circ p, \end{aligned}$$

hence $\beta \circ \tilde{f} \circ \alpha^{-1}$ is indeed a lift of f.

(iv) Let $x \in X$ and $\tilde{x} \in p^{-1}(x)$. Take $\tilde{y} = \tilde{f}(\tilde{x})$ and $\tilde{y}' = \tilde{f}'(\tilde{x})$. By property (i) there exists a $\gamma \in \mathcal{D}(X)$ such that $\gamma(\tilde{y}) = \tilde{y}'$. Thus $(\gamma \circ \tilde{f})(\tilde{x}) = \gamma(\tilde{y}) = \tilde{y}'$. But $\tilde{f}'(\tilde{x}) = \tilde{y}'$ as well. By the uniqueness in property (ii) we obtain that $\gamma \circ \tilde{f} = \tilde{f}'$. □

If we fix a reference lift $\tilde{f}_0$ of f, then for any other lift $\tilde{f}$ there is a unique covering transformation $\gamma \in \mathcal{D}(X)$ such that $\tilde{f} = \gamma \circ \tilde{f}_0$. This provides a one-to-one correspondence between the set of lifts of f and the group of covering transformations $\mathcal{D}(X)$.

We are interested in fixed points of self-maps. The following proposition tells us how fixed points behave under lifts.

Proposition 2.2.7. *Let $f : X \to X$ be a self-map on X, let $p : \tilde{X} \to X$ be the universal cover of X and let $x \in X$.*

(i) Let $\tilde{x} \in p^{-1}(x)$. Then $f(x) = x \iff \tilde{f}(\tilde{x}) \in p^{-1}(x)$.

(ii) $f(x) = x \iff$ for any $\tilde{x} \in p^{-1}(x)$, there is a unique lifting $\tilde{f}$ which leaves $\tilde{x}$ fixed.

(iii) Let $\tilde{f}$ be a lift of f and $\tilde{x} \in p^{-1}(x)$, such that $\tilde{f}(\tilde{x}) = \tilde{x}$. Suppose that $\gamma \in \mathcal{D}(X)$. Then $\gamma \circ \tilde{f} \circ \gamma^{-1}$ is the unique lift of f that has a fixed point at $\gamma(\tilde{x}) \in p^{-1}(x)$.

Proof. We prove the three statements one by one.

(i) If $f(x) = x$, then $(p \circ \tilde{f})(\tilde{x}) = (f \circ p)(\tilde{x}) = f(x) = x$ hence $\tilde{f}(\tilde{x}) \in p^{-1}(x)$. Conversely, $f(x) = (f \circ p)(\tilde{x}) = (p \circ \tilde{f})(\tilde{x}) = x$.

(ii) This follows immediately from combining property (i) with proposition 2.2.6 (ii).

(iii) This follows from the previous properties and proposition 2.2.6. □

This leads us to the following definition:

Definition 2.2.8. Two lifts $\tilde{f}_1$ and $\tilde{f}_2$ of a self-map $f : X \to X$ are *Reidemeister-equivalent* if and only if there exists some $\gamma \in \mathcal{D}(X)$ such that $\tilde{f}_1 = \gamma \circ \tilde{f}_2 \circ \gamma^{-1}$. The equivalence classes (denoted by $[\tilde{f}]$) are called *lifting classes*, and the number of such classes is called the *Reidemeister number* $R(f)$.

There is a close connection between these lifting classes and the fixed points of a self-map.

Theorem 2.2.9. *Let $f : X \to X$ be a self-map, $p : \tilde{X} \to X$ be the universal cover of X and $\tilde{f}, \tilde{f}'$ be lifts of f. Then:*

(i) $\mathrm{Fix}(f) = \bigcup_{\tilde{f}} p(\mathrm{Fix}(\tilde{f}))$,

(ii) $[\tilde{f}] = [\tilde{f}'] \implies p(\mathrm{Fix}(\tilde{f})) = p(\mathrm{Fix}(\tilde{f}'))$,

(iii) $[\tilde{f}] \neq [\tilde{f}'] \implies p(\mathrm{Fix}(\tilde{f})) \cap p(\mathrm{Fix}(\tilde{f}')) = \varnothing$.

Proof. All of this follows from proposition 2.2.7 (ii) and (iii). □

This leads naturally to the definition of (Nielsen) fixed point classes, which were introduced by Jakob Nielsen [Nie24; Nie27].

Definition 2.2.10. The subset $p(\mathrm{Fix}(\tilde{f}))$ of $\mathrm{Fix}(f)$ is called a *fixed point class* of f, determined by the lifting class $[\tilde{f}]$ of f. The number of fixed point classes is called the *Reidemeister number* of f, denoted by $R(f)$.

Of course, we had already defined the Reidemeister number as the number of lifting classes of f. But since the number of lifting classes is the same as the number of fixed point classes, both definitions are equivalent.

We may restate theorem 2.2.9 as follows:

Theorem 2.2.11. *Let $f : X \to X$ be continuous. Then*

$$\mathrm{Fix}(f) = \bigsqcup_{[\tilde{f}]} p(\mathrm{Fix}(\tilde{f})).$$

It may be tempting to interpret this theorem as saying that the fixed point classes form a partition of the set of fixed points. However, there is a small but important caveat: some fixed point classes may be empty. It is imperative to note that an empty fixed point class is still counted as a fixed point class, i.e. the Reidemeister number also counts the empty fixed point classes.

The following theorem provides a more intuitive understanding of what it means for two fixed points to belong to the same fixed point class.

Theorem 2.2.12. *Let $f : X \to X$ be a self-map, $p : \tilde{X} \to X$ be the universal cover of X and $x, x' \in \mathrm{Fix}(f)$. Then x and x' belong to the same fixed point class if and only if there exists a path c in X from x to x' such that*

$$f \circ c \simeq_p c,$$

i.e. c is path-homotopic to its image under f.

Proof. Let x be a fixed point of f that belongs to the fixed point class $p(\mathrm{Fix}(\tilde{f}))$ of a lift $\tilde{f}$. Then there exists some $\tilde{x} \in p^{-1}(x)$ such that $\tilde{f}(\tilde{x}) = \tilde{x}$. Let x' be another fixed point of f.

First, let us assume that x and x' belong to the same fixed point class, i.e. $x' \in p(\mathrm{Fix}(\tilde{f}))$. Take $\tilde{x}' \in p^{-1}(x')$ such that $\tilde{f}(\tilde{x}') = \tilde{x}'$ and choose some path $\tilde{c}$ in $\tilde{X}$ from $\tilde{x}$ to $\tilde{x}'$. Then the path $\tilde{f} \circ \tilde{c}$ is also a path from $\tilde{x}$ to $\tilde{x}'$ and is path-homotopic to $\tilde{c}$, since $\tilde{X}$ is simply connected. Let $c = p \circ \tilde{c}$, which is then a path from x to x' in X. We have that

$$c = p \circ \tilde{c} \simeq_p p \circ (\tilde{f} \circ \tilde{c}) = (p \circ \tilde{f}) \circ \tilde{c} = (f \circ p) \circ \tilde{c} = f \circ c.$$

Conversely, let c be a path from x to x' which is path-homotopic to its image under f, and let $\tilde{c}$ be the lift of c that starts at $\tilde{x}$. Since $c \simeq_p f \circ c$, we must have that $\tilde{c}$ and $\tilde{f} \circ \tilde{c}$ both end in the same terminal point $\tilde{x}' \in p^{-1}(x')$. Hence $\tilde{f}(\tilde{x}') = \tilde{x}'$, thus $x' \in p(\mathrm{Fix}(\tilde{f}))$. ☐

At this point, one may wonder why the theorem above is not used as the definition of a fixed point class. The reason is that this theorem does not account for the existence of empty fixed point classes, and as we will show later, the number of non-empty fixed point classes is not a homotopy invariant. The number of all fixed point classes, including the empty ones, is.

Theorem 2.2.13. *The Reidemeister number $R(f)$ is a homotopy invariant.*

Proof. Let $f, g : X \to X$ be self-maps and let $H = \{h_t\}_{t \in I} : X \times I \to X$ be a homotopy between f and g. Since $p \times \mathrm{id} : \tilde{X} \times I \to X \times I$ is the universal cover of $X \times I$, we may consider lifts $\tilde{H} = \{\tilde{h}_t\}_{t \in I}$ of H such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} \times I & \xrightarrow{\tilde{H}} & \tilde{X} \\ \downarrow{\scriptstyle p\times \mathrm{id}} & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{H} & X \end{array}$$

This means that for every $t \in I$, $\tilde{h}_t$ must be a lift of h_t. Because lifts are unique once we fix a pair of points, $\tilde{H}$ is uniquely determined once we fix $\tilde{h}_0 = \tilde{f}$, and in particular $\tilde{h}_1 = \tilde{g}$ will be uniquely determined. The same reasoning can be done for the inverse homotopy $H^{-1} = \{h_{1-t}\}_{t\in I}$, hence H induces a one-to-one correspondence between the lifts of f and g.

Now, if $\tilde{H} = \{\tilde{h}_t\}_{t\in I}$ is a homotopy between $\tilde{f}$ and $\tilde{g}$, then for every $\gamma \in \mathcal{D}(X)$ we have that $\{\gamma \circ \tilde{h}_t \circ \gamma^{-1}\}_{t\in I}$ is a homotopy between $\gamma \circ \tilde{f} \circ \gamma^{-1}$ and $\gamma \circ \tilde{g} \circ \gamma^{-1}$. Thus, the one-to-one correspondence preserves the lifting classes and hence $R(f) = R(g)$. □

The example below illustrates that, while the Reidemeister number is invariant under homotopies, the (non)-emptiness of a fixed point class is not.

Example 2.2.14. Consider the unit circle S^1, and represent points of the circle by the angle $\theta \in [0, 2\pi)$. The universal cover of S^1 is given by

$$p : \mathbb{R} \to S^1 : t \mapsto t \bmod 2\pi.$$

Consider the map

$$f : S^1 \to S^1 : \theta \mapsto \theta + \varepsilon \bmod 2\pi,$$

where ε is small. Then f is homotopic to id_{S^1}, and an explicit homotopy is given by

$$H = \{h_t\}_{t\in I} : S^1 \times I \to S^1 : (\theta, t) \mapsto \theta + t\varepsilon \bmod 2\pi.$$

If we take the lift

$$\tilde{H} = \{\tilde{h}_t\}_{t\in I} : \mathbb{R} \times I \to \mathbb{R} : (\theta, t) \mapsto \theta + t\varepsilon,$$

we find that $p(\mathrm{Fix}(\tilde{h}_0)) = S^1$ but $p(\mathrm{Fix}(\tilde{h}_1)) = \varnothing$.

We conclude this section by discussing the topological properties of fixed point classes, which will be important for the next section on fixed point indices.

Proposition 2.2.15. *Every fixed point class $\mathbb{F}$ of a self-map f is open and closed in* $\mathrm{Fix}(f)$.

Proof. We will start by proving that a fixed point class $\mathbb{F}$ is open in $\mathrm{Fix}(f)$. Let $x \in \mathbb{F}$ and let V be a neighbourhood of x such that every loop with base point x is path-homotopic (in X) to the trivial loop at x. Now take an open neighbourhood $U_x \subseteq V \cap f^{-1}(V)$ of x that is path-connected. We claim that

$$\mathbb{F} = \left(\bigcup_{x\in\mathbb{F}} U_x\right) \cap \mathrm{Fix}(f) = \bigcup_{x\in\mathbb{F}} \left(U_x \cap \mathrm{Fix}(f)\right).$$

It suffices to prove that if $y \in U_x \cap \mathrm{Fix}(f)$ for some $x \in \mathbb{F}$, then $y \in \mathbb{F}$. Because U_x is path-connected, there exists some path c in $U_x \subseteq V$ from x to y. Then $f \circ c$ is a path from x to y in V as well. By the definition of V, c and $f \circ c$ are path-homotopic in X, and by theorem 2.2.12 x and y belong to the same fixed point class.

Next, we prove that a fixed point class $\mathbb{F}$ is closed in $\mathrm{Fix}(f)$. Consider the open set U defined as

$$U := \bigcup_{x\in\mathrm{Fix}(f)\setminus\mathbb{F}} U_x.$$

Now $\mathbb{F}$ is exactly the intersection of $\mathrm{Fix}(f)$ and the complement of U. □

Proposition 2.2.16. *If X is Hausdorff, then $\mathrm{Fix}(f)$ is closed in X for any self-map $f : X \to X$.*

Proof. Since X is Hausdorff, the diagonal $\Delta_X := \{(x, x) \mid x \in X\}$ is a closed subset of $X \times X$. But $\mathrm{Fix}(f) = (f \times \mathrm{id})^{-1}(\Delta_X)$. □

Corollary 2.2.17. *If X is compact and Hausdorff, then the number of non-empty fixed point classes of a self-map $f : X \to X$ is finite.*

Proof. For any $x \in \mathrm{Fix}(f)$, let the set U_x be as in the proof of proposition 2.2.15. For any fixed point class $\mathbb{F}$, we define the open set $U_\mathbb{F}$ as

$$U_\mathbb{F} := \bigcup_{x\in\mathbb{F}} U_x.$$

Consider the open cover of X given by $X \setminus \mathrm{Fix}(f)$ and all the open sets $U_\mathbb{F}$. By compactness of X, this covering must have a finite subcover, hence only finitely many of the sets $U_\mathbb{F}$ can be non-empty. □

2.3 Fixed point index

In the previous section, we have introduced fixed point classes, and we have seen that the emptiness of a fixed point class is not a homotopy invariant. Moreover, a fixed point class could also contain more than a single point, meaning that so far these fixed point classes tell us very little about the number of fixed points.

In order to study fixed point classes in more detail, we will assign an integer to each of them, called the fixed point index. We will take the approach from [FPS04], which is to introduce this index as the unique function satisfying certain properties, though we will also briefly mention the original construction.

Throughout this section, let X be a connected, Hausdorff, second-countable smooth manifold.

Definition 2.3.1. Let U be an open subset of X. The pair (f, U) is called *admissible* if $\mathrm{Fix}(f, U) := \mathrm{Fix}(f) \cap U$ is compact in X. Denote the set of all admissible pairs on X by $\mathcal{A}(X)$.

Let $H : X \times I \to X$ be a homotopy with $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for all $x \in X$. We call H *admissible* in U if the set

$$\{(x, t) \in U \times I \mid H(x, t) = x\}$$

is compact in $X \times I$.

We can now define the fixed point index.

Theorem 2.3.2. *There exists a (unique) function* $\mathrm{Ind} : \mathcal{A}(X) \to \mathbb{Z}$ *satisfying the following:*

Weak Normalisation
If $f : X \to X$ *is a constant function, then*

$$\mathrm{Ind}(f, X) = 1.$$

Additivity
Let (f, U) *be an admissible pair and* U_1, U_2 *disjoint subsets of* U *such that* $\mathrm{Fix}(f, U) \subseteq U_1 \cup U_2$*. Then*

$$\mathrm{Ind}(f, U) = \mathrm{Ind}(f, U_1) + \mathrm{Ind}(f, U_2).$$

Homotopy-invariance
If H *is an admissible homotopy on* U *between functions* f_0 *and* f_1*, then*

$$\mathrm{Ind}(f_0, U) = \mathrm{Ind}(f_1, U).$$

Proposition 2.3.3. *The function* Ind *as defined above satisfies the following properties:*

Empty set
$\operatorname{Ind}(f, \varnothing) = 0$.

Solution
If (f, U) *is admissible and* $\operatorname{Ind}(f, U) \neq 0$*, then* $\operatorname{Fix}(f, U) \neq \varnothing$*.*

Excision
If V *is an open subset of* U *containing* $\operatorname{Fix}(f, U)$*, then*

$$\operatorname{Ind}(f, U) = \operatorname{Ind}(f, V).$$

Commutativity
Let X, Y *be connected manifolds and* $f : X \to Y$*,* $g : Y \to X$ *be continuous maps such that* $(g \circ f, U)$ *is admissible. Then* $(f \circ g, g^{-1}(U))$ *is admissible as well, and*

$$\operatorname{Ind}(g \circ f, U) = \operatorname{Ind}(f \circ g, g^{-1}(U)).$$

Strong normalisation
Let X *be compact, then*

$$\operatorname{Ind}(f, X) = L(f).$$

We can now define a (more or less) new function Ind which assigns integers to pairs consisting of a function and a fixed point class, rather than to admissible pairs. However, we will need to assume that the space X is compact.

Any fixed point class $\mathbb{F}$ is an open subset of $\operatorname{Fix}(f)$, hence there exists an open $U \subseteq X$ such that $\mathbb{F} = \operatorname{Fix}(f) \cap U$. At the same time, $\mathbb{F}$ is closed in $\operatorname{Fix}(f)$ and hence in X, and therefore it is compact (by compactness of X). We then define

$$\operatorname{Ind}(f, \mathbb{F}) := \operatorname{Ind}(f, U).$$

Note that this definition is independent of the choice of open set U. Indeed, if V is another open subset of X such that $\mathbb{F} = \operatorname{Fix}(f) \cap V$, then also $\mathbb{F} = \operatorname{Fix}(f) \cap (U \cap V)$, and by the excision property we have

$$\operatorname{Ind}(f, U) = \operatorname{Ind}(f, U \cap V) = \operatorname{Ind}(f, V).$$

The original construction of the fixed point index used the degree of maps on spheres, see [Bro71; Jia83]. For completeness' sake, we will give the construction below. Let $x \in \operatorname{Fix}(f)$ be an isolated fixed point. Because X has a manifold structure, there exists an open neighbourhood $U \ni x$ such that

- $U, f(U)$ lie completely in the image of a chart $\psi : V \subseteq \mathbb{R}^n \to X$,

- $U, f(U)$ do not contain any other fixed points.

Then $f' := \psi^{-1} \circ f \circ \psi$ is a function $\psi^{-1}(U) \subseteq \mathbb{R}^n \to \mathbb{R}^n$. Now pick a sphere $S^{n-1}_{\psi^{-1}(x)} \subseteq \psi^{-1}(U)$ centred around $\psi^{-1}(x)$, and define

$$\phi_f : S^{n-1}_{\psi^{-1}(x)} \to S^{n-1} : y \mapsto \frac{y - f'(y)}{\|y - f'(y)\|}.$$

The index $\mathrm{Ind}(f, x)$ of the fixed point x is then defined as the degree of this map.

We conclude this section with the Lefschetz-Hopf fixed point theorem, which follows from the additivity and strong normalisation properties of the fixed point index.

Theorem 2.3.4 (Lefschetz-Hopf fixed point theorem)**.** *Let $f : X \to X$ be a self-map on a compact space X. Then*

$$L(f) = \sum_{\mathbb{F}} \mathrm{Ind}(f, \mathbb{F}).$$

2.4 The Nielsen number

Because the fixed point index has the solution property, it allows us to make statements about the existence of fixed points. We can now define the Nielsen number $N(f)$ of a self-map f which, unlike $L(f)$ and $R(f)$, tells us something about the number of fixed points.

Definition 2.4.1. A fixed point class $\mathbb{F}$ is called *essential* if $\mathrm{Ind}(f, \mathbb{F}) \neq 0$, and *inessential* otherwise. The *Nielsen number* $N(f)$ of f is the number of essential fixed point classes.

Since every essential fixed point class must contain at least one fixed point (the solution property), we have the following theorem:

Theorem 2.4.2. *The Nielsen number $N(f)$ is a lower bound for the number of fixed points of f, i.e. $N(f) \leq \mathrm{Fix}(f)$.*

By corollary 2.2.17, the number of non-empty fixed point classes is necessarily finite, and thus the number of essential fixed point classes is finite as well.

Proposition 2.4.3. *The Nielsen number is finite.*

We have shown before that the number of (non)-empty fixed point classes is not a homotopy invariant. However, it can be shown that the number of (in)essential fixed point classes is a homotopy invariant.

Theorem 2.4.4. *The Nielsen number is a homotopy invariant, i.e. if $f \simeq g$, then $N(f) = N(g)$.*

The proof is non-trivial and can be found in e.g. [Jia83]. We will give a basic idea of the proof.

Idea of proof. Let $H = \{h_t\}_{t \in I} : X \times I \to X$ be a homotopy with $h_0 = f$ and $h_1 = g$, and let $\mathbb{F}_0$, $\mathbb{F}_1$ be fixed point classes of f and g respectively corresponding through H. We will extend H to the *fat homotopy*

$$H_{fat} : X \times I \to X \times I : (x, t) \mapsto (H(x), t).$$

Because H preserves the lifting classes, there exists a fixed point class $\mathbb{F}$ of H_{fat} such that $\mathbb{F}_0$ and $\mathbb{F}_1$ are exactly the $0-$ and $1-$slices of $\mathbb{F}$.

We know there exists some open set $U \subseteq X \times I$ such that $\mathbb{F} = \mathrm{Fix}(H_{fat}) \cap U$. Define U_t and $\mathbb{F}_t$ as the t-slices of U and $\mathbb{F}$ respectively, then $\mathbb{F}_t = \mathrm{Fix}(h_t) \cap U_t$ is a fixed point class of h_t. By "squeezing H into a thin map resembling h_t" (see [Jia83, Corollary 3.10]) we obtain that

$$\mathrm{Ind}(h_t, \mathbb{F}_t) = \mathrm{Ind}(H_{fat}, \mathbb{F}) \text{ for all } t \in I.$$

In particular, setting $t = 0, 1$, we find that

$$\mathrm{Ind}(f, \mathbb{F}_0) = \mathrm{Ind}(H_{fat}, \mathbb{F}) = \mathrm{Ind}(g, \mathbb{F}_1).$$

Thus, homotopies preserve the index of a fixed point class, and in particular whether or not a fixed point class is essential. □

Corollary 2.4.5. *The Nielsen number $N(f)$ is a lower bound for the number of fixed points of every $g \simeq f$:*

$$N(f) \leq \min_{g \simeq f} \# \mathrm{Fix}(g).$$

Often, the Nielsen number is a sharp bound, as proved by Wecken [Wec42].

Theorem 2.4.6. *Let $f : X \to X$ be a continuous self-map on a compact, connected manifold X with $\dim(X) \geq 3$. Then there exists a self-map $g \simeq f$ such that*

$$N(f) = N(g) = \# \mathrm{Fix}(g).$$

2.5 The Reidemeister number

In the previous sections, we defined an equivalence relation on the set of lifts of a self-map $f : X \to X$. Moreover, we showed there is a one-to-one correspondence between the set of lifts and the group of covering transformations $\mathcal{D}(X)$. Hence, this equivalence relation induces induces an equivalence relation on $\mathcal{D}(X)$ using this one-to-one correspondence. This approach was first suggested by Kurt Reidemeister in [Rei36].

Lemma 2.5.1. *Fix a reference lift $\tilde{f}_0$ of a self-map $f : X \to X$. Then $\tilde{f}_0$ induces an endomorphism f_* on $\mathcal{D}(X)$ given by*

$$f_*(\gamma) \circ \tilde{f}_0 = \tilde{f}_0 \circ \gamma$$

for all $\gamma \in \mathcal{D}(X)$. The choice of reference lift $\tilde{f}_0$ determines f_* up to an inner automorphism of $\mathcal{D}(X)$.

Proof. The existence of f_* is given by part (iv) of proposition 2.2.6. Now let $\tilde{f}_0$ and $\tilde{f}_0'$ be two reference lifts with induced maps f_*, f_*' respectively. Then for some $\alpha \in \mathcal{D}(X)$, we have $\tilde{f}_0' = \alpha \circ \tilde{f}_0$. For every $\gamma \in \mathcal{D}(X)$ we then have that

$$f_*'(\gamma) \circ \tilde{f}_0' = \tilde{f}_0' \circ \gamma$$

and thus

$$f_*'(\gamma) \circ \alpha \circ \tilde{f}_0 = \alpha \circ \tilde{f}_0 \circ \gamma.$$

Rearranging this slightly, we obtain

$$\alpha^{-1} \circ f_*'(\gamma) \circ \alpha \circ \tilde{f}_0 = \tilde{f}_0 \circ \gamma.$$

Thus, for every $\gamma \in \mathcal{D}(X)$ we have $\alpha^{-1} \circ f_*'(\gamma) \circ \alpha = f_*(\gamma)$, and hence $f_*' = \iota_\alpha f_*$, where $\iota_\alpha \in \mathrm{Inn}(\mathcal{D}(X))$ is the inner automorphism

$$\iota_\alpha : \mathcal{D}(X) \to \mathcal{D}(X) : \gamma \mapsto \alpha \circ \gamma \circ \alpha^{-1}.$$ □

There is a natural link between f_π and any f_* defined from lifts. To make this more concrete, recall the group isomorphism between the covering transformations and the fundamental group we mentioned in proposition 2.2.6:

$$\Phi_{\tilde{x}} : \mathcal{D}(X) \to \pi_1(X, x) : \gamma \mapsto [\alpha],$$

which depends on the choice of $\tilde{x} \in p^{-1}(x)$.

Proposition 2.5.2. *Let $\tilde{f}$ be any lift of f, choose $\tilde{x} \in p^{-1}(x)$ and set $\tilde{y} = \tilde{f}(\tilde{x})$. Now let f_* be the endomorphism on $\mathcal{D}(X)$ induced by $\tilde{f}$. Then*

$$\Phi_{\tilde{y}} \circ f_* = f_\pi \circ \Phi_{\tilde{x}},$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}(X) & \xrightarrow{f_*} & \mathcal{D}(X) \\ \cong \downarrow \Phi_{\tilde{x}} & & \cong \downarrow \Phi_{\tilde{y}} \\ \pi_1(X, x) & \xrightarrow{f_\pi} & \pi_1(X, y) \end{array}$$

Proof. Let $\gamma \in \mathcal{D}(X)$. Its image under $\Phi_{\tilde{x}}$ is $[\alpha]$, where

- $\alpha = p \circ \tilde{\alpha}$ for some path $\tilde{\alpha}$ in $\tilde{X}$,
- $\tilde{\alpha}(0) = \tilde{x}$,
- $\tilde{\alpha}(1) = \gamma(\tilde{x})$.

Now, $(f_\pi \circ \Phi_{\tilde{x}})(\gamma) = f_\pi([\alpha]) = [f \circ \alpha]$, and this path satisfies the following:

- $f \circ \alpha = f \circ p \circ \tilde{\alpha} = p \circ \tilde{f} \circ \tilde{\alpha}$,
- $(\tilde{f} \circ \tilde{\alpha})(0) = \tilde{f}(\tilde{x}) = \tilde{y}$,
- $(\tilde{f} \circ \tilde{\alpha})(1) = (\tilde{f} \circ \gamma)(\tilde{x}) = (f_*(\gamma) \circ \tilde{f})(\tilde{x}) = f_*(\gamma)(\tilde{y})$.

On the other hand, $(\Phi_{\tilde{y}} \circ f_*)(\gamma) = \Phi_{\tilde{y}}(f_*(\gamma)) = [\beta]$, where

- $\beta = p \circ \tilde{\beta}$ for some path $\tilde{\beta}$ in $\tilde{X}$,
- $\tilde{\beta}(0) = \tilde{y}$,
- $\tilde{\beta}(1) = f_*(\gamma)(\tilde{y})$.

Thus, $\tilde{f} \circ \tilde{\alpha}$ and $\tilde{\beta}$ are two paths in $\tilde{X}$ from $\tilde{y}$ to $f_*(\gamma)(\tilde{y})$, and must therefore be path-homotopic, and hence $[f \circ \alpha] = [\beta]$. □

Definition 2.5.3. Two elements $\alpha, \beta \in \mathcal{D}(X)$ are *f_*-twisted conjugate* if and only if there exists $\gamma \in \mathcal{D}(X)$ such that

$$\alpha = \gamma \beta f_*(\gamma)^{-1}.$$

Just like the usual notion of conjugacy, this is an equivalence relation. The number of equivalence classes is called the *Reidemeister number* $R(f_*)$.

We show that this is indeed the equivalence relation induced by the equivalence relation on lifts.

Proposition 2.5.4. *Let $\tilde{f}_1, \tilde{f}_2$ be two lifts of a self-map $f : X \to X$ and $\tilde{f}_0$ be a reference lift such that $\tilde{f}_i = \alpha_i \circ \tilde{f}_0$ for $i = 1, 2$, with $\alpha_i \in \mathcal{D}(X)$, and let f_* be the endomorphism on $\mathcal{D}(X)$ induced by $\tilde{f}_0$. Then*

$$\tilde{f}_1 \sim \tilde{f}_2 \iff \alpha_1 \sim_{f_*} \alpha_2,$$

and hence $R(f) = R(f_*)$.

Proof.

$$\begin{aligned}
\tilde{f}_1 \sim \tilde{f}_2 &\iff \alpha_1 \circ \tilde{f}_0 \sim \alpha_2 \circ \tilde{f}_0 \\
&\iff \exists \gamma \in \mathcal{D}(X) : \alpha_1 \circ \tilde{f}_0 = \gamma \circ \alpha_2 \circ \tilde{f}_0 \circ \gamma^{-1} \\
&\iff \exists \gamma \in \mathcal{D}(X) : \alpha_1 \circ \tilde{f}_0 = \gamma \circ \alpha_2 \circ f_*(\gamma)^{-1} \circ \tilde{f}_0 \\
&\iff \exists \gamma \in \mathcal{D}(X) : \alpha_1 = \gamma \circ \alpha_2 \circ f_*(\gamma)^{-1} \\
&\iff \alpha_1 \sim_{f_*} \alpha_2.
\end{aligned}$$

□

2.5.1 Group-theoretic Reidemeister number

Definition 2.5.3 gives a purely algebraic definition of the Reidemeister number. In fact, there is no real reason to only define the Reidemeister number for endomorphisms on fundamental groups induced by self-maps.

Definition 2.5.5. Let G be a group and $\varphi : G \to G$ an endomorphism. Define an equivalence relation $\sim_\varphi$ on G by

$$\forall g, g' \in G : g \sim_\varphi g' \iff \exists h \in G : g = hg'\varphi(h)^{-1}.$$

The equivalence classes are called *Reidemeister classes* or *twisted conjugacy classes*, and we will denote the Reidemeister class of g under the endomorphism φ by $[g]_\varphi$. The set of Reidemeister classes of φ is denoted by $\mathfrak{R}(\varphi)$. The *Reidemeister number* $R(\varphi)$ is the cardinality of $\mathfrak{R}(\varphi)$ and is therefore always a positive integer or infinity.

Definition 2.5.6. Let $\operatorname{Aut}(G)$ be the automorphism group of a group G. We define the *Reidemeister spectrum* as

$$\operatorname{Spec}_R(G) = \{R(\varphi) \mid \varphi \in \operatorname{Aut}(G)\}.$$

If $\operatorname{Spec}_R(G) = \{\infty\}$ we say that G has the R_∞*-property*, and if $\operatorname{Spec}_R(G) = \mathbb{N} \cup \{\infty\}$ we say G has *full* Reidemeister spectrum.

Similarly, we can define such spectrum for endomorphisms:

Definition 2.5.7. Let $\mathrm{End}(G)$ be the set of endomorphisms of a group G. We define the *extended Reidemeister spectrum* as

$$\mathrm{ESpec}_R(G) = \{R(\varphi) \mid \varphi \in \mathrm{End}(G)\}.$$

If $\mathrm{ESpec}_R(G) = \mathbb{N} \cup \{\infty\}$ we say G has *full* extended Reidemeister spectrum.

For any group G, $1 \in \mathrm{ESpec}_R(G)$, since the Reidemeister number of the trivial endomorphism $g \mapsto 1$ is 1. Let us provide some examples.

Example 2.5.8. Let G be a finite abelian group and $\varphi \in \mathrm{End}(G)$. Then for any two elements $g, g' \in G$, we have

$$\begin{aligned} g \sim_\varphi g' &\iff \exists h \in G : g = h + g' - \varphi(h) \\ &\iff \exists h \in G : g - g' = (\mathrm{id} - \varphi)(h) \\ &\iff g - g' \in \mathrm{im}(\mathrm{id} - \varphi). \end{aligned}$$

Thus, for the Reidemeister number $R(\varphi)$ we find that

$$\begin{aligned} R(\varphi) &= \#(G/\mathrm{im}(\mathrm{id} - \varphi)) \\ &= \#G/\#\,\mathrm{im}(\mathrm{id} - \varphi) \\ &= \#\ker(\mathrm{id} - \varphi) \\ &= \#\,\mathrm{Fix}(\varphi). \end{aligned}$$

If $G = \mathbb{Z}_p$ with $p > 2$ prime, then any endomorphism φ is completely determined by the image of 1. We have three cases:

(1) $\varphi(1) = 0$. This is the trivial endomorphism, which only fixes 0, hence $R(\varphi) = 1$.

(2) $\varphi(1) = 1$. This is the identity, which fixes every element, hence $R(\varphi) = p$.

(3) $\varphi(1) = k$, with $1 < k < p$. If $\varphi(x) = x$, then $(k-1)x \equiv 0 \bmod p$, hence either $x = 0$ or $k = 1$. Since we excluded the latter case, 0 is the only fixed point and $R(\varphi) = 1$.

Thus, we have that $\mathrm{Spec}_R(\mathbb{Z}_p) = \mathrm{ESpec}_R(\mathbb{Z}_p) = \{1, p\}$.

To simplify notation in the next example (and the remainder of this thesis), we introduce the following map:

$$|\,.\,|_\infty : \mathbb{Z} \to \mathbb{N} \cup \{\infty\} : x \mapsto |x|_\infty := \begin{cases} |x| & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$$

Example 2.5.9 (see [Bro+75])**.** Let $G = \mathbb{Z}^n$ and $D \in \mathbb{Z}^{n\times n}$ an endomorphism. Just like in the previous example, $R(\varphi) = \#(\mathbb{Z}^n / \operatorname{im}(\mathbb{1}_n - D))$. The Smith normal form of $\mathbb{1}_n - D$ is a diagonal matrix with $a_1, a_2, \ldots, a_k, 0, \ldots, 0$ on the diagonal for some $k \leq n$, $a_i \neq 0$. Then

$$\mathbb{Z}^n / \operatorname{im}(\mathbb{1}_n - D) \cong \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \oplus \cdots \oplus \mathbb{Z}_{a_k} \oplus \mathbb{Z}^{n-k},$$

and hence $R(D) = |\det(\mathbb{1}_n - D)|_\infty$.

The following lemma is pivotal in determining the R_∞-property of many groups.

Lemma 2.5.10 (see [Hea85, Theorem 1.8], [KLL05, §2], [GW09, Lemma 1.1])**.** *Let N be a normal subgroup of a group G and $\varphi \in \operatorname{End}(G)$ with $\varphi(N) \subseteq N$. We denote the restriction of φ to N by $\varphi|_N$, and the induced endomorphism on the quotient G/N by φ'. We then get the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{i} & G & \xrightarrow{p} & G/N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \varphi|_N} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} & & \\ 1 & \longrightarrow & N & \xrightarrow{i} & G & \xrightarrow{p} & G/N & \longrightarrow & 1 \end{array}$$

Note that, if φ and $\varphi|_N$ are both automorphisms, then φ' is an automorphism as well. This diagram induces the following exact sequence of pointed sets:

$$1 \longrightarrow \operatorname{Fix}(\varphi|_N) \xrightarrow{i_{\mathrm{Fix}}} \operatorname{Fix}(\varphi) \xrightarrow{p_{\mathrm{Fix}}} \operatorname{Fix}(\varphi') \xrightarrow{\delta} \mathfrak{R}(\varphi|_N) \xrightarrow{\hat{\imath}} \mathfrak{R}(\varphi) \xrightarrow{\hat{p}} \mathfrak{R}(\varphi') \longrightarrow 1$$

where all maps are evident except δ, which is defined as $\delta(gN) = [g\varphi(g)^{-1}]_{\varphi|_N}$. We obtain the following properties:

(1) $R(\varphi) \geq R(\varphi')$,

(2) if $R(\varphi|_N) = \infty$ *and* $|\operatorname{Fix}(\varphi')| < \infty$*, then* $R(\varphi) = \infty$*,*

(3) if $R(\varphi|_N) < \infty$*,* $R(\varphi') < \infty$ *and* $N \subseteq Z(G)$*, then* $R(\varphi) \leq R(\varphi|_N)R(\varphi')$*.*

Proof. Proving that the diagram commutes and that it has exact rows is straightforward. For the cohomological background of the exact sequence, we refer to [FT15, Section 2.2].

The only map that is not obviously well-defined, is δ. Consider the natural action of $\operatorname{Fix}(\varphi')$ on $\mathfrak{R}(\varphi|_N)$ given by

$$gN \cdot [n]_{\varphi|_N} = [gn\varphi(g)^{-1}]_{\varphi|_N}.$$

Since

$$p(gn\varphi(g)^{-1}) = gN \cdot \varphi'(gN)^{-1} = 1N,$$

we have that $gn\varphi(g)^{-1} \in N$. Moreover, if $g'N = gN$, then $g = n'g'$ for some $n' \in N$. Hence

$$[gn\varphi(g)^{-1}]_{\varphi|_N} = [n'g'n\varphi(g')^{-1}\varphi|_N(n')^{-1}]_{\varphi|_N} = [g'n\varphi(g')^{-1}]_{\varphi|_N},$$

so this action is well-defined. We can write δ in terms of this action as $\delta(gN) = gN \cdot [1]_{\varphi|_N}$, hence δ is well-defined.

Next, we prove the exactness of the sequence step by step.

(1) i_{Fix} is injective and $\operatorname{im}(i_{\mathrm{Fix}}) = \ker(p_{\mathrm{Fix}})$. These follow readily from the exactness of the diagram.

(2) $\operatorname{im}(p_{\mathrm{Fix}}) \subseteq \ker(\delta)$. Let $gN \in \operatorname{im}(p_{\mathrm{Fix}})$, then we may assume that $\varphi(g) = g$. So

$$\delta(gN) = [g\varphi(g)^{-1}]_{\varphi|_N} = [gg^{-1}]_{\varphi|_N} = [1]_{\varphi|_N},$$

therefore $gN \in \ker(\delta)$.

(3) $\operatorname{im}(p_{\mathrm{Fix}}) \supseteq \ker(\delta)$. Let $gN \in \ker(\delta)$, then $[g\varphi(g)^{-1}]_{\varphi|_N} = [1]_{\varphi|_N}$. This means there exists some $n \in N$ such that $ng\varphi(g)^{-1}\varphi|_N(n)^{-1} = 1$, which is equivalent to $ng = \varphi(ng)$. Then $ng \in \operatorname{Fix}(\varphi)$ and $p_{\mathrm{Fix}}(ng) = gN$, thus $gN \in \operatorname{im}(p_{\mathrm{Fix}})$.

(4) $\operatorname{im}(\delta) \subseteq \ker(\hat{\imath})$. Let $[n]_{\varphi|_N} \in \operatorname{im}(\delta)$, hence there exists $gN \in \operatorname{Fix}(\varphi')$ such that $[n]_{\varphi|_N} = [g\varphi(g)^{-1}]_{\varphi|_N}$. Then

$$\hat{\imath}([n]_{\varphi|_N}) = \hat{\imath}([g\varphi(g)^{-1}]_{\varphi|_N}) = [g\varphi(g)^{-1}]_{\varphi} = [1]_{\varphi},$$

thus $[n]_{\varphi|_N} \in \ker(\hat{\imath})$.

(5) $\mathrm{im}(\delta) \supseteq \ker(\hat{\imath})$. Let $[n]_{\varphi|_N} \in \ker(\hat{\imath})$, then

$$\hat{\imath}([n]_{\varphi|_N}) = [n]_\varphi = [1]_\varphi,$$

so there exists some $g \in G$ such that $g\varphi(g)^{-1} = n$, or equivalently $\varphi(g) = n^{-1}g$. Then $\varphi'(gN) = gN$ and

$$\delta(gN) = [g\varphi(g)^{-1}]_{\varphi|_N} = [n]_{\varphi|_N},$$

therefore $[n]_{\varphi|_N} \in \mathrm{im}(\delta)$.

(6) $\mathrm{im}(\hat{\imath}) = \ker(\hat{p})$ and $\hat{p}$ is surjective. Again, these follow readily from the exactness of the diagram.

Finally, we prove the 3 properties:

(1) This follows from $\hat{p}$ being surjective,

(2) Since $\mathfrak{R}(\varphi|_N)$ is infinite and $\mathrm{Fix}(\varphi')$ is finite, the action of $\mathrm{Fix}(\varphi')$ on $\mathfrak{R}(\varphi|_N)$ divides the latter into infinitely many orbits. However, two elements $[n]_{\varphi|_N}$ and $[n]_{\varphi|_N}$ belong to the same orbit if and only if $i(n) \sim_\varphi i(n')$, thus $\mathfrak{R}(\varphi)$ must be infinite.

(3) Let

$$\mathfrak{R}(\varphi|_N) = \{[n_1]_{\varphi|_N}, [n_2]_{\varphi|_N}, \dots, [n_{R(\varphi|_N)}]_{\varphi|_N}\},$$

$$\mathfrak{R}(\varphi') = \{[g_1N]_{\varphi'}, [g_2N]_{\varphi'}, \dots, [g_{R(\varphi')}N]_{\varphi'}\}.$$

Let $g \in G$, then $gN \in [g_iN]_{\varphi'}$ for some i, so there exists some $hN \in G/N$ such that

$$gN = hN \cdot g_iN \cdot \varphi'(hN)^{-1} = hg_i\varphi(h)^{-1}N.$$

Hence there exists some $n \in N$ such that

$$g = hg_i\varphi(h)^{-1}n.$$

In turn, $n \in [n_j]_{\varphi|_N}$ for some j, hence there exists an $m \in N$ such that

$$n = mn_j\varphi|_N(m)^{-1}.$$

Since $n, m \in N \subseteq Z(G)$, we obtain

$$g = hg_i\varphi(h)^{-1}mn_j\varphi|_N(m)^{-1} = (hm)(g_in_j)\varphi(hm)^{-1},$$

therefore $g \in [g_in_j]_\varphi$. Since this is true for arbitrary $g \in G$, we obtain that $R(\varphi) \leq R(\varphi|_N)R(\varphi')$. □

Note that in the above lemma, if the group G is abelian, then all of the sets of Reidemeister classes are abelian groups as well, and the exact sequence becomes an exact sequence of groups. This is used in the following example.

Example 2.5.11. Let G be a finitely generated abelian group and $\varphi \in \mathrm{End}(G)$. Then $G = \mathbb{Z}^n \oplus \tau(G)$ where $\tau(G)$ is the (fully characteristic) torsion subgroup, hence we get the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \tau(G) & \xrightarrow{i} & G & \xrightarrow{p} & \mathbb{Z}^n & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \varphi|_{\tau(G)}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} & & \\ 1 & \longrightarrow & \tau(G) & \xrightarrow{i} & G & \xrightarrow{p} & \mathbb{Z}^n & \longrightarrow & 1 \end{array}$$

The induced endomorphism φ' on $G/\tau(G) \cong \mathbb{Z}^n$ is given by some matrix $D \in \mathbb{Z}^{n\times n}$. Since $R(\varphi') = |\det(\mathbb{1}_n - D)|_\infty$, we have that

$$R(\varphi') < \infty \iff \det(\mathbb{1}_n - D) \neq 0 \iff \mathrm{Fix}(\varphi') = 1.$$

First, consider the case where $R(\varphi') = \infty$. Then $R(\varphi) = \infty$ as well due to the first property in lemma 2.5.10. Second, let $R(\varphi') < \infty$. Then the exact sequence becomes

$$1 \longrightarrow \mathfrak{R}(\varphi|_{\tau(G)}) \xrightarrow{\hat{\imath}} \mathfrak{R}(\varphi) \xrightarrow{\hat{p}} \mathfrak{R}(\varphi') \longrightarrow 1$$

However, since G is abelian, all of these sets of Reidemeister classes inherit the (abelian) group law from G, and hence this is an exact sequence of finite groups. In particular, we have that

$$R(\varphi) = R(\varphi|_{\tau(G)})R(\varphi').$$

The same result actually holds for any group of the form $G = \mathbb{Z}^n \oplus \tau(G)$ with $\tau(G)$ any finite (not necessarily abelian) group, see [Fel00, Proposition 3].

Corollary 2.5.12. *Let N be a characteristic subgroup of G. If either*

(1) the quotient G/N has the R_∞-property, or

(2) N has finite index in G and has the R_∞-property,

then G has the R_∞-property as well.

Proof. This follows from property 1 in lemma 2.5.10. □

The following proposition is well-known, see for example [KLL05, Lemma 2.1]. However, identity (3) is usually proven only for an endomorphism f_* on a fundamental group induced by a self-map f, since the proof is topological in nature. We will provide a purely group-theoretic proof.

Proposition 2.5.13. *Consider the situation from lemma 2.5.10. Let $g \in G$ and ι_g its corresponding inner automorphism. Then we have the following exact sequence of pointed sets*

$$1 \longrightarrow \operatorname{Fix}(\iota_g\varphi|_N) \xrightarrow{i_g} \operatorname{Fix}(\iota_g\varphi) \xrightarrow{p_g} \operatorname{Fix}(\iota_{gN}\varphi') \xrightarrow{\delta_g} \mathfrak{R}(\iota_g\varphi|_N) \xrightarrow{\hat{\imath}_g} \mathfrak{R}(\iota_g\varphi) \xrightarrow{\hat{p}_g} \mathfrak{R}(\iota_{gN}\varphi') \longrightarrow 1$$

from which we obtain the following identities:

(1) $\#\hat{p}^{-1}([gN]_{\varphi'}) = \#\operatorname{im}(\hat{\imath}_g)$,

(2) $R(\varphi) = \sum_{[gN]_{\varphi'}} \#\operatorname{im}(\hat{\imath}_g)$,

(3) $\#\hat{\imath}_g^{-1}([n]_{\iota_g\varphi}) = [\operatorname{Fix}(\iota_{gN}\varphi') : p_{ng}(\operatorname{Fix}(\iota_{ng}\varphi))]$,

(4) $[G : N] = \#[gN]_{\iota_{gN}\varphi'} \cdot \#\operatorname{Fix}(\iota_{gN}\varphi')$.

Proof. We will prove this item per item. When $g = 1$, the maps $\hat{\imath}_1$ and $\hat{p}_1$ equal the maps $\hat{\imath}$ and $\hat{p}$ from lemma 2.5.10 respectively.

(1) First, note that $\#\hat{p}^{-1}([gN]_{\varphi'}) = \#\hat{p}_g^{-1}([1N]_{\iota_{gN}\varphi'})$ because

$$\begin{aligned}
[h]_\varphi \in \hat{p}^{-1}([gN]_{\varphi'}) &\iff [hN]_{\varphi'} = [gN]_{\varphi'} \\
&\iff \exists k \in G : hN = kg\varphi(k)^{-1}N \\
&\iff \exists k \in G : hg^{-1}N = kg\varphi(k)^{-1}g^{-1}N \\
&\iff [hg^{-1}N]_{\iota_{gN}\varphi'} = [1N]_{\iota_{gN}\varphi'} \\
&\iff [hg^{-1}]_{\iota_g\varphi} \in \hat{p}_g^{-1}([1N]_{\iota_{gN}\varphi'}).
\end{aligned}$$

By exactness, $\hat{p}_g^{-1}([1N]_{\iota_{gN}\varphi'}) = \operatorname{im}(\hat{\imath}_g)$.

(2) Because $\hat{p}$ is surjective, we have the disjoint union

$$\mathfrak{R}(\varphi) = \bigsqcup_{[gN]_{\varphi'}} \hat{p}^{-1}([gN]_{\varphi'}).$$

Applying (1), we get

$$R(\varphi) = \sum_{[gN]_{\varphi'}} \#\hat{p}^{-1}([gN]_{\varphi'}) = \sum_{[gN]_{\varphi'}} \#\operatorname{im}(\hat{\imath}_g).$$

(3) $\#\hat{\imath}_g^{-1}([n]_{\iota_g\varphi}) = \#\hat{\imath}_{ng}^{-1}([1]_{\iota_{ng}\varphi})$, because

$$\begin{aligned}
[m]_{\iota_g\varphi|_N} \in \hat{\imath}_g^{-1}([n]_{\iota_g\varphi}) &\iff [m]_{\iota_g\varphi} = [n]_{\iota_g\varphi} \\
&\iff \exists k \in G : m = kn(\iota_g\varphi)(k)^{-1} \\
&\iff \exists k \in G : mn^{-1} = kn(\iota_g\varphi)(k)^{-1}n^{-1} \\
&\iff [mn^{-1}]_{\iota_{ng}\varphi} = [1]_{\iota_{ng}\varphi} \\
&\iff [mn^{-1}]_{\iota_{ng}\varphi|_N} \in \hat{\imath}_{ng}^{-1}([1]_{\iota_{ng}\varphi}).
\end{aligned}$$

By exactness, $\hat{\imath}_{ng}^{-1}([1]_{\iota_{ng}\varphi}) = \operatorname{im}(\delta_{ng})$. Now note that

$$\begin{aligned}
\delta_{ng}(h_1N) = \delta_{ng}(h_2N) &\iff [h_1(\iota_{ng}\varphi)(h_1)^{-1}]_{\iota_{ng}\varphi|_N} = \\
&\qquad [h_2(\iota_{ng}\varphi)(h_2)^{-1}]_{\iota_{ng}\varphi|_N} \\
&\iff \exists m \in N : h_1(\iota_{ng}\varphi)(h_1)^{-1} = \\
&\qquad mh_2(\iota_{ng}\varphi)(h_2)^{-1}(\iota_{ng}\varphi)(m)^{-1} \\
&\iff \exists m \in N : h_2^{-1}m^{-1}h_1 \in \operatorname{Fix}(\iota_{ng}\varphi) \\
&\iff h_2^{-1}h_1N \in p_{ng}(\operatorname{Fix}(\iota_{ng}\varphi)).
\end{aligned}$$

Therefore, $\#\operatorname{im}(\delta_{ng}) = [\operatorname{Fix}(\iota_{gN}\varphi') : p_{ng}(\operatorname{Fix}(\iota_{ng}\varphi))]$, from which the result follows.

(4) The quotient group G/N acts transitively on $[gN]_{\iota_{gN}\varphi'}$ by

$$hN \cdot [gN]_{\iota_{gN}\varphi'} = [hg\varphi(h)^{-1}N]_{\iota_{gN}\varphi'}.$$

The result then follows from the orbit-stabiliser theorem. □

We can now prove the following inequalities.

Proposition 2.5.14. *Let G be a group with finite index normal subgroup N, and φ an endomorphism such that $\varphi(N) \subseteq N$. Then we have*

$$\sum_{gN} R(\iota_g\varphi|_N) \geq R(\varphi) \geq \frac{1}{[G:N]} \sum_{gN} R(\iota_g\varphi|_N).$$

Proof. From proposition 2.5.13 we can deduce that

$$\begin{aligned}
R(\varphi) &= \sum_{[gN]_{\varphi'}} \# \operatorname{im}(\hat{\imath}_g) \\
&= \sum_{[gN]_{\varphi'}} \sum_{[n]_{\iota_g\varphi} \in \operatorname{im}(\hat{\imath}_g)} 1 \\
&= \sum_{gN} \frac{1}{\#[gN]_{\varphi'}} \sum_{[n]_{\iota_g\varphi|_N}} \frac{1}{\#\hat{\imath}_g^{-1}([n]_{\iota_g\varphi})} \qquad (2.1) \\
&= \sum_{gN} \sum_{[n]_{\iota_g\varphi|_N}} \frac{\#p_{ng}(\operatorname{Fix}(\iota_{ng}\varphi))}{\#[gN]_{\varphi'} \cdot \# \operatorname{Fix}(\iota_{gN}\varphi')} \\
&= \frac{1}{[G:N]} \sum_{gN} \sum_{[n]_{\iota_g\varphi|_N}} \#p_{ng}(\operatorname{Fix}(\iota_{ng}\varphi)).
\end{aligned}$$

Clearly $\#p_{ng}(\operatorname{Fix}(\iota_{ng}\varphi)) \geq 1$, hence

$$R(\varphi) \geq \frac{1}{[G:N]} \sum_{gN} \sum_{[n]_{\iota_g\varphi|_N}} 1 = \frac{1}{[G:N]} \sum_{gN} R(\iota_g\varphi|_N).$$

On the other hand, $\#p_{ng}(\operatorname{Fix}(\iota_{ng}\varphi)) \leq [G:N]$, hence

$$R(\varphi) \leq \frac{1}{[G:N]} \sum_{gN} \sum_{[n]_{\iota_g\varphi|_N}} [G:N] = \sum_{gN} R(\iota_g\varphi|_N).$$ □

Corollary 2.5.15. *Let G be a group with finite index normal subgroup N, and φ an endomorphism such that $\varphi(N) \subseteq N$. Then we have*

$$R(\varphi) = \infty \iff \exists gN \in G/N \text{ such that } R(\iota_g\varphi|_N) = \infty.$$

The following formula is often called the *averaging formula.*

Proposition 2.5.16. *Let G be a torsion-free group with finite index normal subgroup N; and let φ be an endomorphism such that $\varphi(N) \subseteq N$ and $\operatorname{Fix}(\iota_g \varphi|_N) = 1$ for all $g \in G$. Then we have*

$$R(\varphi) = \frac{1}{[G : N]} \sum_{gN} R(\iota_g \varphi|_N).$$

Proof. Consider eq. (2.1): it suffices to show that $\operatorname{Fix}(\iota_g \varphi) = 1$ for all $g \in G$. Suppose that h is a fixed point of $\iota_g \varphi$. Because G/N is finite, there exists some $k \in \mathbb{N}$ such that $h^k N = 1N$, and therefore $h^k \in N$. But then $(\iota_g \varphi|_N)(h^k) = (\iota_g \varphi)(h)^k = h^k$. Because 1 is the only fixed point of $\iota_g \varphi|_N$, this means that $h^k = 1$, and because G is torsion-free, $h = 1$. □

A similar result is the following, often called the *addition formula*.

Proposition 2.5.17 (see [Won01, Proposition 1])**.** *Let G be a group with finite index normal subgroup N; and let φ be an endomorphism such that $\varphi(N) \subseteq N$ and $\operatorname{Fix}(\iota_{gN} \varphi') = 1$ for all $g \in G$. Then we have*

$$R(\varphi) = \sum_{[gN]_{\varphi'}} R(\iota_g \varphi|_N).$$

Proof. Consider proposition 2.5.13(2). If $\operatorname{Fix}(\iota_{gN} \varphi') = 1$ for all $g \in G$, then $\# \operatorname{im}(\hat{\imath}_g) = R(\iota_g \varphi|_N)$. □

Lemma 2.5.18. *Let $G = G_1 \times G_2$ be a direct product where both $G_1 \times \{1\}$ and $\{1\} \times G_2$ are (fully) characteristic subgroups. Then $\operatorname{Aut}(G) \cong \operatorname{Aut}(G_1) \times \operatorname{Aut}(G_2)$ ($\operatorname{End}(G) \cong \operatorname{End}(G_1) \times \operatorname{End}(G_2)$), and for any automorphism (endomorphism) $\varphi = \varphi_1 \times \varphi_2$ we have $R(\varphi) = R(\varphi_1)R(\varphi_2)$. Hence $\operatorname{Spec}_R(G) = \operatorname{Spec}_R(G_1) \cdot \operatorname{Spec}_R(G_2)$ ($\operatorname{ESpec}_R(G) = \operatorname{ESpec}_R(G_1) \cdot \operatorname{ESpec}_R(G_2)$).*

Proof. It is straightforward to work out that the map

$$\mathfrak{R}(\varphi) \to \mathfrak{R}(\varphi_1) \times \mathfrak{R}(\varphi_2) : [(g_1, g_2)]_\varphi \mapsto ([g_1]_{\varphi_1}, [g_2]_{\varphi_2})$$

is a bijection. The result follows immediately. □

We give a lemma that gives equality of Reidemeister numbers of different endomorphisms of the same group.

Lemma 2.5.19 (see [FLT08, Corollary 3.2])**.** *Let G be a group and let $\varphi_1, \varphi_2 \in \operatorname{End}(G)$. If either of the following holds:*

(1) $\exists \iota \in \mathrm{Inn}(G)$ *such that* $\varphi_1 = \varphi_2 \circ \iota$*,*

(2) $\exists \psi \in \mathrm{Aut}(G)$ *such that* $\varphi_1 = \psi \circ \varphi_2 \circ \psi^{-1}$*,*

then $R(\varphi_1) = R(\varphi_2)$*.*

Proof. We will prove this case by case.

(1) There exists some $g \in G$ such that $\iota(h) = ghg^{-1}$ for all $g \in G$. Let $x \sim_{\varphi_1} y$, then there exists $z \in G$ such that

$$x = zy\varphi_1(z)^{-1} = zy\varphi_2(gzg^{-1})^{-1} = zg^{-1}gy\varphi_2(gzg^{-1})^{-1}.$$

Multiplying on the left by g we get

$$gx = \left(gzg^{-1}\right)(gy)\,\varphi_2(gzg^{-1})^{-1},$$

hence the map

$$\hat{g} : \mathfrak{R}(\varphi_1) \to \mathfrak{R}(\varphi_2) : [x]_{\varphi_1} \mapsto [gx]_{\varphi_2}$$

is a well-defined bijection, and $R(\varphi_1) = R(\varphi_2)$.

(2) Let $x \sim_{\varphi_1} y$, then there exists $z \in G$ such that

$$x = zy\varphi_1(z)^{-1} = zy(\psi \circ \varphi_2 \circ \psi^{-1})(z)^{-1}.$$

Applying ψ^{-1} to both sides gives us

$$\psi^{-1}(x) = \psi^{-1}(z)\psi^{-1}(y)\varphi_2(\psi^{-1}(z))^{-1},$$

thus $\psi^{-1}(x) \sim_{\varphi_2} \psi^{-1}(y)$. Hence the map

$$\hat{\psi} : \mathfrak{R}(\varphi_1) \to \mathfrak{R}(\varphi_2) : [x]_{\varphi_1} \mapsto [\psi^{-1}(x)]_{\varphi_2}$$

is a well-defined bijection, and then $R(\varphi_1) = R(\varphi_2)$. □

2.6 Dynamical zeta functions

Inspired by the Hasse-Weil zeta function of an algebraic variety over a finite field, in [AM65] Artin and Mazur defined the zeta function of a map $f : X \to X$ on a topological space X as

$$F_f(z) := \exp \sum_{n=1}^{\infty} \frac{F(f^n)}{n} z^n,$$

where $F(f^n)$ is the number of isolated fixed points of f^n. For axiom A diffeomorphisms on a compact manifold, this zeta function was shown to be rational [Man71], while in general it need not be, see for example [BL70].

Definition 2.6.1. Let us use $\zeta_a(z)$ to denote a zeta function of the form

$$\zeta_a(z) := \exp \sum_{n=1}^{\infty} \frac{a_n}{n} z^n.$$

We say that $\zeta_a(z)$ is determined by the sequence $a = (a_n)_{n\in\mathbb{N}}$.

We are particularly interested in when a zeta function $\zeta_a(z)$ is rational, since this means that the infinite sequence of coefficients $(a_n)_{n\in\mathbb{N}}$ is determined by a finite set of complex numbers, i.e. the zeroes and poles of $\zeta_a(z)$.

When we say that a zeta function is rational, we actually mean that there exists a positive radius of convergence on which the power series converges to a rational function. The power series

$$\sum_{n=1}^{\infty} \frac{b^n}{n} z^n = -\log(1 - bz),$$

with $b \in \mathbb{C}$, has radius of convergence $1/|b|$ if $b \neq 0$ and converges on the entire complex plane otherwise. The following lemma makes use of this to explicitly give a link between the rationality and the zeroes and poles.

Lemma 2.6.2. *A zeta function $\zeta_a(z)$ is a rational function if and only if there exist complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_k, \mu_1, \mu_2, \ldots, \mu_l$ such that*

$$a_n = \sum_{j=1}^{l} \mu_j^n - \sum_{i=1}^{k} \lambda_i^n$$

for all $n \in \mathbb{N}$. In particular, the numbers $1/\lambda_i$ and $1/\mu_j$ are exactly the zeroes and poles of $\zeta_a(z)$ respectively.

Proof. If $\zeta_a(z)$ is a rational function, it is of the form

$$\zeta_a(z) = \frac{\prod_{i=1}^{k}(1 - \lambda_i z)}{\prod_{j=1}^{l}(1 - \mu_j z)},$$

since $\zeta_a(0) = 1$. Taking the logarithmic derivative, we get

$$\begin{aligned}
\frac{d}{dz}\log\zeta_a(z) &= \frac{d}{dz}\log\frac{\prod_{i=1}^{k}(1-\lambda_i z)}{\prod_{j=1}^{l}(1-\mu_j z)} \\
&= \frac{d}{dz}\left(\sum_{i=1}^{k}\log(1-\lambda_i z) - \sum_{j=1}^{l}\log(1-\mu_j z)\right) \\
&= -\sum_{i=1}^{k}\frac{\lambda_i}{1-\lambda_i z} + \sum_{j=1}^{l}\frac{\mu_i}{1-\mu_i z} \\
&= -\sum_{i=1}^{k}\sum_{n=1}^{\infty}\lambda_i^n z^{n-1} + \sum_{j=1}^{l}\sum_{n=1}^{\infty}\mu_i^n z^{n-1} \\
&= \sum_{n=1}^{\infty}\left(\sum_{j=1}^{l}\mu_i^n - \sum_{i=1}^{k}\lambda_i^n\right) z^{n-1},
\end{aligned}$$

which must equal

$$\frac{d}{dz}\log\zeta_a(z) = \frac{d}{dz}\sum_{n=1}^{\infty}\frac{a_n}{n}z^n = \sum_{n=1}^{\infty}a_n z^{n-1}.$$

The converse follows from a direct calculation. In particular, the radius of convergence r is given by

$$r = \frac{1}{\max\{|\lambda_1|,\ldots,|\lambda_k|,|\mu_1|,\ldots,|\mu_l|\}}. \qquad \square$$

It is easy to prove the following corollary using this lemma.

Corollary 2.6.3. *Consider two zeta functions*

$$\zeta_a(z) = \exp\sum_{n=1}^{\infty}\frac{a_n}{n}z^n, \qquad \xi_b(z) = \exp\sum_{n=1}^{\infty}\frac{b_n}{n}z^n,$$

and their additive convolution

$$(\zeta_a * \xi_b)(z) := \exp\sum_{n=1}^{\infty}\frac{a_n b_n}{n}z^n.$$

If ζ_a and ξ_b are rational, then so is $\zeta_a * \xi_b$.

Inspired by the Artin-Mazur zeta function, Smale introduced the *Lefschetz zeta function* [Sma67] as

$$L_f(z) := \exp \sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n.$$

Smale immediately proved the following:

Theorem 2.6.4. *Let $f : X \to X$ be a map on a compact polyhedron X. Then the Lefschetz zeta function L_f of f is rational.*

Proof. We know that

$$L(f^n) = \sum_{i=0}^{\dim X} (-1)^i \operatorname{tr}(f^n_{i,*} : H_i(X, \mathbb{Q}) \to H_i(X, \mathbb{Q})).$$

Since the homology groups $H_i(X, \mathbb{Q})$ are finite-dimensional vector spaces over $\mathbb{Q}$ and the $f_{*,i}$ are linear maps, we may express the Lefschetz numbers $L(f^n)$ in terms of the eigenvalues of $f_{i,*}$:

$$L(f^n) = \sum_k a_k^n - \sum_l b_l^n,$$

where $a_k, b_l \in \mathbb{C}$ are the eigenvalues. The result now follows from lemma 2.6.2. □

Fel'shtyn defined the *Nielsen zeta function* of a self-map f analogously as

$$N_f(z) := \exp \sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n,$$

and proved that it has positive radius of convergence [Fel88; FP85]. Unlike the Lefschetz zeta function, the Nielsen function need not be rational in general. The question of whether or not a Nielsen zeta function is rational has been studied recently in various papers, e.g. [DD15; DTV18; Fel01; Li94; Rom11].

In [Fel91], Fel'shtyn defined the Reidemeister zeta function of a self-map f as

$$R_f(z) := \exp \sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n.$$

He also defined the same Reidemeister zeta function for Reidemeister numbers of group endomorphisms.

Definition 2.6.5. Let φ be an endomorphism of a group G, such that $R(\varphi^n) < \infty$ for all $n \in \mathbb{N}$. Then we can define the *Reidemeister zeta function* of φ as

$$R_\varphi(z) := \exp \sum_{n=1}^{\infty} \frac{R(\varphi^n)}{n} z^n.$$

Note that a Reidemeister zeta function of a self-map f or an endomorphism φ only exists if the Reidemeister numbers $R(f^n)$ or $R(\varphi^n)$ are finite for all $n \in \mathbb{N}$.

Example 2.6.6 (see [FH94, Lemma 5])**.** Let G be a finite abelian group and $\varphi \in \operatorname{End}(G)$. We call $g \in G$ *periodic* if there exists some $k \in \mathbb{N}$ such that $\varphi^k(g) = g$, and we define its *φ-periodic orbit* by

$$[g] := \{g, \varphi(g), \dots, \varphi^{k-1}(g)\}.$$

In example 2.5.8, we have shown that $R(\varphi^n) = \#\operatorname{Fix}(\varphi^n)$. But an element $g \in G$ is a fixed point of φ^n if and only if it is periodic and $\#[g]|n$, hence

$$R(\varphi^n) = \sum_{\substack{[g] \\ \#[g]|n}} \#[g].$$

We can then calculate the Reidemeister zeta function of φ as

$$\begin{aligned}
R_\varphi(z) &= \exp \sum_{n=1}^{\infty} \sum_{\substack{[g] \\ \#[g]|n}} \frac{\#[g]}{n} z^n \\
&= \exp \sum_{[g]} \sum_{n=1}^{\infty} \frac{\#[g]}{\#[g]n} z^{\#[g]n} \\
&= \prod_{[g]} \exp \sum_{n=1}^{\infty} \frac{1}{n} z^{\#[g]n} \\
&= \prod_{[g]} \exp \left(-\log(1 - z^{\#[g]}) \right) \\
&= \prod_{[g]} \frac{1}{1 - z^{\#[g]}},
\end{aligned}$$

which is rational and has radius of convergence 1.

Example 2.6.7 (see [Fel00, Lemma 15])**.** Let $G = \mathbb{Z}^n$ for some $n \geq 1$, and let $D \in \mathbb{Z}^{n\times n}$ be an endomorphism. We know from example 2.5.9 that $R(D) = |\det(\mathbb{1}_n - D)|_\infty$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of D, then

$$|\det(\mathbb{1}_n - D^k)| = \prod_{i=1}^{n} |1 - \lambda_i^k|.$$

We now consider 4 cases:

1. $\lambda_i \in \mathbb{R}$ and $|\lambda_i| < 1$. Then $|1 - \lambda_i^k| = 1^k - \lambda_i^k$.
2. $\lambda_i \in \mathbb{R}$ and $\lambda_i < -1$. Then $|1 - \lambda_i^k| = -(-1)^k + (-\lambda_i)^k$.
3. $\lambda_i \in \mathbb{R}$ and $\lambda_i > 1$. Then $|1 - \lambda_i^k| = -1^k + \lambda_i^k$.
4. $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$. Then its complex conjugate $\bar{\lambda}_i$ is an eigenvalue of D as well, and
$$|1 - \lambda_i^k||1 - \bar{\lambda}_i^k| = 1^k - \lambda_i^k - \bar{\lambda}_i^k + |\lambda_i|^{2k}.$$

Thus, the product $\prod_{i=1}^{n} |1 - \lambda_i^k|$ can be expanded as a sum of terms of the form $\pm \left(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_p}\right)^k$ with $p \in \{0, 1, \dots, n\}$. For the sake of brevity, we write

$$\prod_{i=1}^{n} |1 - \lambda_i^k| = \sum_{i=1}^{a} \mu_i^k - \sum_{j=1}^{b} \nu_j^k$$

for certain $\mu_i, \nu_j \in \mathbb{C}$. We can then calculate the Reidemeister zeta function of φ as

$$\begin{aligned} R_\varphi(z) &= \exp \sum_{n=1}^{\infty} \left(\sum_{i=1}^{a} \mu_i^k - \sum_{j=1}^{b} \nu_j^k \right) \frac{1}{n} z^n \\ &= \exp \left(\sum_{i=1}^{a} \sum_{n=1}^{\infty} \frac{\mu_i^k}{n} z^n - \sum_{j=1}^{b} \sum_{n=1}^{\infty} \frac{\nu_j^k}{n} z^n \right) \\ &= \exp \left(-\sum_{i=1}^{a} \log(1 - \mu_i z) + \sum_{j=1}^{b} \log(1 - \nu_j z) \right) \\ &= \frac{\prod_{j=1}^{b} (1 - \nu_j z)}{\prod_{i=1}^{a} (1 - \mu_i z)}, \end{aligned}$$

which is rational and has radius of convergence r given by

$$r = \frac{1}{\max\{|\mu_1|, \dots, |\mu_a|, |\nu_1|, \dots, |\nu_b|\}}.$$

Applying corollary 2.6.3 to the previous two examples, we also find the following. This result was obtained through different means in [Fel91, Theorem 2].

Example 2.6.8. Let $G = \mathbb{Z}^n \oplus \tau(G)$ be a finitely generated abelian group and $\varphi \in \mathrm{End}(G)$. From example 2.5.11, we know that $R(\varphi) = R(\varphi|_{\tau(G)})R(\varphi')$, with φ' the induced endomorphism on $G/\tau(G) \cong \mathbb{Z}^n$. But then $R_\varphi(z)$ is exactly the convolution $R_{\varphi|_{\tau(G)}}(z) * R_{\varphi'}(z)$ of two rational functions, which must be rational by corollary 2.6.3.

Combining corollary 2.6.3 with lemma 2.5.18, we obtain:

Corollary 2.6.9. *Let $G = G_1 \times G_2$ be a direct product of groups, and consider an endomorphism φ of the form $\varphi = \varphi_1 \times \varphi_2$. If $R_{\varphi_1}(z)$ and $R_{\varphi_2}(z)$ are rational, then so is $R_\varphi(z)$.*

Finally, let us end this chapter with some examples about the existence of Reidemeister zeta functions:

Example 2.6.10. Let $G = \mathbb{Z}$, whose endomorphisms φ_m are completely determined by $\varphi_m(1) = m$. From example 2.6.7, we can see that the Reidemeister zeta function $R_{\varphi_m}(z)$ will exist if and only if $m \notin \{-1, 1\}$, or in other words when φ_m is not an automorphism. We separate three cases:

- $m = 0$, then $R_{\varphi_m}(z) = \frac{1}{1-z}$;
- $m < -1$, then $R_{\varphi_m}(z) = \frac{1+z}{1+mz}$;
- $m > 1$, then $R_{\varphi_m}(z) = \frac{1-z}{1-mz}$.

Example 2.6.11. Let $G = \mathbb{Z}^n$ for $n \geq 2$. From example 2.6.7 we can see that the Reidemeister zeta function $R_D(z)$ of an endomorphism $D \in \mathbb{Z}^{n \times n}$ will exist if and only if D has no roots of unity as eigenvalues.

In contrast to the one-dimensional case, we have that for any $n \geq 2$ there exists an automorphism $D \in \mathrm{GL}_n(\mathbb{Z})$ such that its Reidemeister zeta function $R_D(z)$ exists. Let $M_2 \in \mathrm{GL}_2(\mathbb{Z})$ and $M_3 \in \mathrm{GL}_3(\mathbb{Z})$ be the matrices

$$M_2 := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_3 := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

neither of which have roots of unity as eigenvalues. Thus, depending on whether n is even or odd, take D to be

$$D = \begin{pmatrix} M_2 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_2 \end{pmatrix} \text{ or } D = \begin{pmatrix} M_2 & & & & \\ & M_2 & & & \\ & & \ddots & & \\ & & & M_2 & \\ & & & & M_3 \end{pmatrix}$$

respectively. Then $R_D(z)$ exists.

Chapter 3

Almost-crystallographic groups and infra-nilorbifolds

Almost-crystallographic groups are generalisations of crystallographic groups, a family of groups well understood by the so-called Bieberbach theorems. The almost-crystallographic groups arise as the fundamental groups of infra-nilorbifolds (almost flat orbifolds), and similarly the torsion-free almost-crystallographic groups, which are also called almost-Bieberbach groups, are the fundamental groups of the infra-nilmanifolds (almost flat manifolds). This allows an algebraic study of these spaces and their topological properties.

For more information on crystallographic groups and almost-crystallographic groups, we refer to [Szc12] and [Dek96] respectively. For information on compact flat manifolds and infra-nilmanifolds we refer to [Cha86] and [Dek18].

3.1 Nilpotent groups

For an arbitrary group G, we can define the k-fold commutator group $\gamma_k(G)$ inductively as

$$\gamma_1(G) := G \quad \text{and} \quad \gamma_{k+1}(G) := [G, \gamma_k(G)].$$

Definition 3.1.1. The *lower central series* of a group G is the series of groups

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_k(G) \geq \cdots$$

If this series eventually becomes the trivial group, i.e. for some $c \in \mathbb{N}$ we have $\gamma_c(G) \neq 1$ and $\gamma_{c+1}(G) = 1$, then we call G *nilpotent*. We say G has *nilpotency class* c, and if $c = 1$ that G is *abelian*.

Example 3.1.2. The *discrete Heisenberg group* $H_3(\mathbb{Z})$ is defined as

$$H_3(\mathbb{Z}) := \langle a, b, c \mid [a, b] = c, [a, c] = 1, [b, c] = 1\rangle.$$

Its lower central series is given by

$$H_3(\mathbb{Z}) \geq \langle c\rangle \geq 1,$$

hence $H_3(\mathbb{Z})$ has nilpotency class 2.

Proposition 3.1.3. *The k-fold commutator groups $\gamma_k(G)$ are fully characteristic subgroups, i.e. for every $\varphi \in \mathrm{End}(G)$, $\varphi(\gamma_k(G)) \subseteq \gamma_k(G)$. Hence any endomorphism (automorphism) of G restricts to an endomorphism (automorphism) of $\gamma_k(G)$.*

Corollary 3.1.4. *Let G be a group and $\varphi \in \mathrm{End}(G)$ ($\mathrm{Aut}(G)$). Then φ induces an endomorphism (automorphism) $(\varphi)_k$ on $\gamma_k(G)/\gamma_{k+1}(G)$.*

While the k-fold commutator groups behave very well under endomorphisms, they will sometimes not have the properties we require. In order to rectify this, we first need to introduce the concept of isolators.

Definition 3.1.5. Let G be a group. For a subgroup $H \leq G$, the *isolator* of H in G is defined as

$$\sqrt[G]{H} = \{g \in G \mid \exists n \in \mathbb{N} : g^n \in H\}.$$

In general, the isolator of a subgroup $H \leq G$ need not be a subgroup itself, for example the isolator of the trivial group is the set of torsion elements $\tau(G)$. The isolators of k-fold commutator subgroups, however, satisfy some nice properties.

Lemma 3.1.6 (see [Dek96, Lemma 1.1.2 and Lemma 1.1.4])**.** *Let G be a group. Then*

(i) $\forall k \in \mathbb{N} : \sqrt[G]{\gamma_k(G)}$ *is a fully characteristic subgroup of G.*

(ii) $\forall k \in \mathbb{N} : G/\sqrt[G]{\gamma_k(G)}$ *is torsion-free.*

(iii) $\forall k, l \in \mathbb{N} : [\sqrt[G]{\gamma_k(G)}, \sqrt[G]{\gamma_l(G)}] \leq \sqrt[G]{\gamma_{k+l}(G)}$.

(iv) $\forall k, l \in \mathbb{N}$ *with $l \geq k$: if $N := \sqrt[G]{\gamma_l(G)}$, then*

$$\sqrt[G/N]{\gamma_k(G/N)} = \sqrt[G]{\gamma_k(G)}/N.$$

These properties allow us to define a new central series composed of isolators of k-fold commutators.

Definition 3.1.7. The *adapted lower central series* of a group G is given by

$$G = \sqrt[G]{\gamma_1(G)} \geq \sqrt[G]{\gamma_2(G)} \geq \cdots \geq \sqrt[G]{\gamma_k(G)} \geq \cdots$$

The adapted lower central series of a group G will eventually terminate if and only if G is a torsion-free, nilpotent group. The main advantage of using this central series over the lower central series is that the factors are torsion-free.

Proposition 3.1.8. *All factors $\sqrt[G]{\gamma_k(G)}/\sqrt[G]{\gamma_{k+1}(G)}$ in the adapted lower central series are torsion-free.*

Proof. First note that $G/\sqrt[G]{\gamma_{k+1}(G)}$ is a torsion-free group for any $k \in \mathbb{N}$. Since $\sqrt[G]{\gamma_k(G)}/\sqrt[G]{\gamma_{k+1}(G)}$ is a subgroup of this group, it is torsion-free as well. □

3.1.1 Finitely generated, torsion-free, nilpotent groups

We are particularly interested in the case where G is a finitely generated, torsion-free, nilpotent group. These groups are the nilpotent generalisations of the free abelian groups $\mathbb{Z}^n$.

Proposition 3.1.9. *Let G be a finitely generated, torsion-free, nilpotent group. Then the factors of the (adapted) lower central series are finitely generated, (torsion-free), abelian groups, i.e. for all $k \in \mathbb{N}$ we have*

$$\frac{\gamma_k(G)}{\gamma_{k+1}(G)} \cong \mathbb{Z}^{n_k} \oplus F_k,$$

$$\frac{\sqrt[G]{\gamma_k(G)}}{\sqrt[G]{\gamma_{k+1}(G)}} \cong \mathbb{Z}^{n_k},$$

for some $n_k \in \mathbb{N}$ and some finite, abelian group F_k.

Example 3.1.10. Fix some $k \in \mathbb{N}$ and consider the finitely generated, torsion-free, nilpotent group

$$N := \langle a, b, c \mid [a, b] = c^k, [a, c] = 1, [b, c] = 1 \rangle.$$

We find that $\gamma_2(N) = \langle c^k \rangle$, $\sqrt[N]{\gamma_2(N)} = \langle c \rangle$ and $\gamma_3(N) = \sqrt[N]{\gamma_3(N)} = 1$. Thus, the lower central series has factors

$$\frac{N}{\gamma_2(N)} \cong \langle a, b, c \mid [a, b] = 1, [a, c] = 1, [b, c] = 1, c^k = 1 \rangle \cong \mathbb{Z}^2 \oplus \mathbb{Z}_k,$$

$$\frac{\gamma_2(N)}{\gamma_3(N)} \cong \langle c^k \rangle \cong \mathbb{Z}.$$

The adapted lower central series, however, has factors

$$\frac{N}{\sqrt[N]{\gamma_2(N)}} \cong \langle a, b \mid [a, b] = 1 \rangle \cong \mathbb{Z}^2,$$

$$\frac{\sqrt[N]{\gamma_2(N)}}{\sqrt[N]{\gamma_3(N)}} \cong \langle c \rangle \cong \mathbb{Z}.$$

Definition 3.1.11. A group G is called *polycyclic* if and only if it admits a series of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_{n-1} \geq G_n = 1,$$

such that $G_{i+1} \triangleleft G_i$ and the factors G_i/G_{i+1} are cyclic. The number of infinite cyclic factors in this series is called the *Hirsch length* $h(G)$ of the group G.

Example 3.1.12. The discrete Heisenberg group

$$H_3(\mathbb{Z}) := \langle a, b, c \mid [a, b] = c, [a, c] = 1, [b, c] = 1 \rangle$$

has the series of subgroups

$$H_3(\mathbb{Z}) \geq \langle b, c \mid [b, c] = 1 \rangle \geq \langle c \rangle \geq 1,$$

for which every factor is isomorphic to $\mathbb{Z}$. Thus $H_3(\mathbb{Z})$ is polycyclic and has Hirsch length 3.

Theorem 3.1.13 (see [KM79, Theorem 17.2.2])**.** *Finitely generated, torsion-free, nilpotent groups are poly-$\mathbb{Z}$, i.e. they are of the form*

$$(((\mathbb{Z} \rtimes \mathbb{Z}) \rtimes \mathbb{Z}) \cdots) \rtimes \mathbb{Z}.$$

Corollary 3.1.14. *A finitely generated, torsion-free, nilpotent group is polycyclic.*

3.1.2 Free nilpotent groups

Definition 3.1.15. The *free nilpotent group* $N_{r,c}$ of rank $r > 1$ and nilpotency class c is the quotient

$$N_{r,c} := \frac{F_r}{\gamma_{c+1}(F_r)},$$

where F_r is the free group on r generators.

We excluded the case $r = 1$ from the definition above. If $r = 1$, then $F_r = \mathbb{Z}$, and hence $\gamma_k(F_r) = 1$ for all $k \geq 2$. Therefore $N_{1,c} = F_1/\gamma_{c+1}(F_1) = F_1$ for all c, which means this group has nilpotency class 1 and not c.

Example 3.1.16. The following are examples of free nilpotent groups.

(1) For any $r \in \mathbb{N}$, the free nilpotent group $N_{r,1}$ is isomorphic to $\mathbb{Z}^r$.

(2) The free nilpotent group $N_{2,2}$ is isomorphic to the discrete Heisenberg group $H_3(\mathbb{Z})$ from example 3.1.12.

Free nilpotent groups have the nice property that their lower central series coincides with their adapted lower central series, hence the factors of the lower central series are torsion-free. The following proposition makes this more exact.

Proposition 3.1.17. *Let $N_{r,c}$ be the free nilpotent group of rank r and nilpotency class c. Then the factors of its lower central series, i.e. the groups*

$$\frac{\gamma_k(N_{r,c})}{\gamma_{k+1}(N_{r,c})}$$

are isomorphic to $\mathbb{Z}^{n_k}$, where

$$n_k = \frac{1}{k}\sum_{d|k} \mu(d) r^{k/d},$$

with μ the Möbius function:

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ 0 & \text{if } d \text{ is not square-free}, \\ (-1)^n & \text{if } d \text{ is the product of } n \text{ distinct primes.} \end{cases} \tag{3.1}$$

We will skip the proof of this proposition, but the formula for n_k will follow from proposition 3.2.29 later in this thesis. In particular, one finds the following:

Corollary 3.1.18. *The Hirsch length of a free nilpotent group $N_{r,c}$ is given by*

$$h(N_{r,c}) = \sum_{k=1}^{c} \frac{1}{k} \sum_{d|k} \mu(d) r^{k/d},$$

3.2 Lie groups and Lie algebras

In this section, we give a concise summary of the theory of Lie groups and Lie algebras, with a focus on nilpotent Lie groups and algebras.

Definition 3.2.1. A *Lie group* G is a smooth manifold equipped with a group structure, such that the maps

$$G \times G \to G : (g_1, g_2) \mapsto g_1 g_2,$$

$$G \to G : g \mapsto g^{-1},$$

are smooth.

Definition 3.2.2. A subgroup H of a Lie group G is called a *Lie subgroup* if it is equipped with a manifold structure that makes it a Lie group and the inclusion map $H \hookrightarrow G$ is an immersion.

Let us consider some standard examples:

Example 3.2.3. The real numbers $\mathbb{R}$ give rise to many examples of Lie groups.

(1) The n-dimensional real space $\mathbb{R}^n$ with addition and its natural manifold structure is a Lie group.

(2) The non-zero real numbers $\mathbb{R}_0$ with multiplication and the positive real numbers $\mathbb{R}^+$ with multiplication both form a Lie group.

(3) $\mathrm{GL}_n(\mathbb{R})$ inherits a manifold structure when seen as a subset of $\mathbb{R}^{n^2}$, and forms a Lie group when we consider matrix multiplication as its operation.

(4) The *Heisenberg group* $H_3(\mathbb{R})$, defined as

$$H_3(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\},$$

is a Lie subgroup of $\mathrm{GL}_3(\mathbb{R})$.

Definition 3.2.4. A *Lie group morphism* $f : G \to H$ is a smooth map that is also a group morphism. If f is bijective and f^{-1} is a Lie group morphism as well, then f is called a *Lie group isomorphism.*

Note that it would suffice to define a Lie group morphism as a continuous group morphism between Lie groups, since any such map is automatically smooth, see for example [Fer98, Theorem 3.7.1].

Example 3.2.5. The following are examples of Lie group morphisms.

(1) If H is a Lie subgroup of a Lie group G, then the inclusion map $H \hookrightarrow G$ is an injective Lie group morphism.

(2) If G is a Lie group and $g \in G$, then the inner automorphism

$$\iota_g : G \to G : h \mapsto ghg^{-1}$$

is a Lie group automorphism.

(3) The determinant map

$$\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}_0 : M \mapsto \det M$$

is a surjective Lie group morphism.

Definition 3.2.6. A *Lie algebra* $\mathfrak{g}$ is a vector space equipped with a bilinear map

$$[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (X, Y) \mapsto [X, Y]$$

called the *Lie bracket*, which satisfies:

ALTERNATIVITY
For all $X \in \mathfrak{g}$:

$$[X, X] = 0.$$

JACOBI IDENTITY
For all $X, Y, Z \in \mathfrak{g}$:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Example 3.2.7. The following are examples of Lie algebras.

(1) Any vector space V becomes an abelian Lie algebra if we equip it with the trivial Lie bracket, i.e. $[X, Y] = 0$ for all $X, Y \in V$, e.g. $\mathbb{R}^n$.

(2) Any associative algebra A becomes a Lie algebra if we equip it with the Lie bracket

$$[a, b] = ab - ba,$$

e.g. the $n \times n$-matrices over the real numbers $\mathbb{R}^{n\times n}$ equipped with this Lie bracket form a Lie algebra, which is usually denoted by $\mathfrak{gl}_n(\mathbb{R})$.

Definition 3.2.8. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras and $\phi : \mathfrak{g} \to \mathfrak{h}$ is a linear map, then ϕ is called a *Lie algebra morphism* if and only if

$$\phi([X, Y]) = [\phi(X), \phi(Y)]$$

for all $X, Y \in \mathfrak{g}$. If ϕ is bijective, it is called a *Lie algebra isomorphism*.

Example 3.2.9. The following are examples of Lie algebra morphisms.

(1) If V, W are vector spaces equipped with the trivial Lie bracket, then any linear map $V \to W$ is a Lie algebra morphism.

(2) The trace map
$$\operatorname{tr} : \mathfrak{gl}_n(\mathbb{R}) \to \mathbb{R} : M \mapsto \operatorname{tr} M$$
is a Lie algebra morphism.

There is a very close connection between Lie groups and Lie algebras. Let G be a Lie group with identity 1_G, and define $\mathfrak{g}$ as the tangent space at 1_G, i.e.
$$\mathfrak{g} := T_{1_G}G.$$
An inner automorphism ι_g of G is a Lie group isomorphism that fixes 1_G, and hence induces an automorphism $(\iota_g)_*$ on $\mathfrak{g}$. Define
$$\operatorname{Ad} : G \to \operatorname{GL}(\mathfrak{g}) : g \mapsto (\iota_g)_*,$$
which is a smooth map and hence induces a linear map between the tangent spaces:
$$\operatorname{Ad}_* : \mathfrak{g} \to T_{\operatorname{id}} \operatorname{GL}(\mathfrak{g}).$$
Now note that $\operatorname{GL}(\mathfrak{g})$ is an open subset of $\Lambda(\mathfrak{g}, \mathfrak{g})$, the vector space of linear self-maps on $\mathfrak{g}$. Thus, we may identify $T_{\operatorname{id}} \operatorname{GL}(\mathfrak{g})$ with $T_{\operatorname{id}}\Lambda(\mathfrak{g}, \mathfrak{g}) = \Lambda(\mathfrak{g}, \mathfrak{g})$. Now define
$$\operatorname{ad} = \operatorname{Ad}_* : \mathfrak{g} \to \Lambda(\mathfrak{g}, \mathfrak{g}).$$

Proposition 3.2.10. *Let G be a Lie group with identity 1_G and $\mathfrak{g}$ its tangent space at 1_G. If we define a Lie bracket by*
$$[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (X, Y) \mapsto [X, Y] = \operatorname{ad}(X)(Y),$$
then $\mathfrak{g}$ equipped with this bracket is a Lie algebra, called the Lie algebra associated to *G.*

Example 3.2.11. The following are examples of Lie algebras associated to Lie groups.

(1) $\mathbb{R}^n$ is the Lie algebra associated to the Lie group $\mathbb{R}^n$.

(2) $\mathbb{R}$ is the Lie algebra associated to both of the Lie groups $\mathbb{R}_0$ and $\mathbb{R}^+$.

(3) $\mathfrak{gl}_n(\mathbb{R})$ is the Lie algebra associated to $\operatorname{GL}_n(\mathbb{R})$.

A Lie group morphism $f : G \to H$ must map 1_G to 1_H, and therefore induces a map $f_* : T_{1_G}G \to T_{1_H}H$. But these tangent spaces are exactly the associated Lie algebras, and f_* will actually be a Lie algebra morphism.

Definition 3.2.12. Let G, H be Lie groups with associated Lie algebras $\mathfrak{g}, \mathfrak{h}$ respectively. A Lie group morphism $f : G \to H$ induces a Lie algebra morphism $f_* : \mathfrak{g} \to \mathfrak{h}$ called the *Lie algebra morphism induced by* f.

Proposition 3.2.13. *Let* G *be a Lie group with associated Lie algebra* $\mathfrak{g}$*. For any* $X \in \mathfrak{g}$*, there exists a unique Lie group morphism* $\varphi_X : \mathbb{R} \to G$ *such that* $(\varphi_X)_*(1) = X$*.*

Definition 3.2.14. Let G be a Lie group with associated Lie algebra $\mathfrak{g}$. We define the *exponential map* as

$$\exp : \mathfrak{g} \to G : X \mapsto \varphi_X(1).$$

Lemma 3.2.15. *Let* G, H *be Lie groups with associated Lie algebras* $\mathfrak{g}, \mathfrak{h}$ *respectively. Let* $f : G \to H$ *be a Lie group morphism inducing a Lie algebra morphism* $f_* : \mathfrak{g} \to \mathfrak{h}$*, then the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{f_*} & \mathfrak{h} \end{array}$$

Example 3.2.16. The following are examples of exponential maps.

(1) The exponential map from the Lie algebra $\mathbb{R}^n$ to the Lie group $\mathbb{R}^n$ is the identity map.

(2) The exponential map from the Lie algebra $\mathbb{R}$ to the Lie group $\mathbb{R}_0$ is the usual exponential map, and similarly for the Lie group $\mathbb{R}^+$.

(3) The exponential map from the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ to the Lie group $\mathrm{GL}_n(\mathbb{R})$ is given by

$$\exp : \mathfrak{gl}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}) : M \mapsto \sum_{i=0}^{\infty} \frac{M^i}{i!}. \tag{3.2}$$

Also note that the induced morphism by the determinant map is exactly the trace map, i.e.

$$\det(\exp M) = \exp(\operatorname{tr} M).$$

(4) The Lie algebra $\mathfrak{h}$ associated to the Heisenberg group $H_3(\mathbb{R})$ defined in example 3.2.3(4) is given by

$$\mathfrak{h} := \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Since $M^3 = 0$ for any matrix $M \in \mathfrak{h}$, the exponential map from eq. (3.2) reduces to

$$\exp : \mathfrak{h} \to H_3(\mathbb{R}) : M \mapsto \mathbb{1} + M + \frac{M^2}{2},$$

or equivalently,

$$\exp : \mathfrak{h} \to H_3(\mathbb{R}) : \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & y + \frac{xz}{2} \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

3.2.1 Nilpotent Lie groups and Lie algebras

We have already defined what a nilpotent group is, hence we can consider nilpotent Lie groups. Let us define nilpotency for a Lie algebra $\mathfrak{g}$ with Lie bracket $[.,.]$. We define $\gamma_k(\mathfrak{g})$ inductively as

$$\gamma_1(\mathfrak{g}) := \mathfrak{g} \quad \text{and} \quad \gamma_{k+1}(\mathfrak{g}) := [\mathfrak{g}, \gamma_k(\mathfrak{g})].$$

Definition 3.2.17. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if its lower central series

$$\mathfrak{g} = \gamma_1(\mathfrak{g}) \geq \gamma_2(\mathfrak{g}) \geq \cdots \geq \gamma_k(\mathfrak{g}) \geq \cdots$$

eventually becomes trivial, i.e. for some $c \in \mathbb{N}$ we have $\gamma_c(\mathfrak{g}) \neq \{0\}$ and $\gamma_{c+1}(\mathfrak{g}) = \{0\}$. We then say $\mathfrak{g}$ has *nilpotency class* c, and if $c = 1$ that $\mathfrak{g}$ is *abelian.*

Clearly, the definitions of nilpotency for a Lie group (definition 3.1.1) and a Lie algebra (definition 3.2.17) are very similar. Knowing the connection between a Lie group and its associated Lie algebra, it is natural to assume that there must be some connection between their respective lower central series. This is indeed the case:

Theorem 3.2.18 (see [Hoc65, Theorem XII.3.1])**.** *Let G be a connected, nilpotent Lie group with associated Lie algebra $\mathfrak{g}$. Every term $\gamma_k(G)$ of the lower central series of G will be a Lie subgroup of G and have $\gamma_k(\mathfrak{g})$ as its associated Lie algebra. Thus, $\mathfrak{g}$ is also nilpotent and its nilpotency class coincides with that of G.*

Let us now focus on connected, simply connected, nilpotent Lie groups.

Theorem 3.2.19. *Let G be a connected, simply connected, nilpotent Lie group with associated Lie algebra $\mathfrak{g}$. Equipping $\mathfrak{g}$ with the natural manifold structure, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism.*

Example 3.2.20. The exponential map from $\mathbb{R}$ to $\mathbb{R}_0$ (which is not connected) is not surjective, however, the exponential map from $\mathbb{R}$ to $\mathbb{R}^+$ is a diffeomorphism.

Definition 3.2.21. Let G be a connected, simply connected, nilpotent Lie group with associated Lie algebra $\mathfrak{g}$. We may define the logarithmic map $\log : G \to \mathfrak{g}$ as the inverse of the exponential map.

Example 3.2.22. The following are examples of logarithmic maps.

(1) The logarithmic map from the Lie group $\mathbb{R}^n$ to the Lie algebra $\mathbb{R}^n$ is the identity map.

(2) The logarithmic map from the Lie group $\mathbb{R}^+$ to the Lie algebra $\mathbb{R}$ is the usual logarithmic map.

(3) The logarithmic map from the Heisenberg group $H_3(\mathbb{R})$ to its associated Lie algebra $\mathfrak{h}$ is given by

$$\log : H_3(\mathbb{R}) \to \mathfrak{h} : M \mapsto (M - \mathbb{1}) - \frac{(M - \mathbb{1})^2}{2},$$

or equivalently,

$$\log : H_3(\mathbb{R}) \to \mathfrak{h} : \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & x & y - \frac{xz}{2} \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

Proposition 3.2.23. *Let G, H be connected, simply connected, nilpotent Lie groups with associated Lie algebras $\mathfrak{g}, \mathfrak{h}$ respectively. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism, then there exists a (unique) Lie group morphism $f : G \to H$ such that $f_* = \phi$.*

3.2.2 Lattices in nilpotent Lie groups

Lattices of connected, simply connected, nilpotent Lie groups are a vital ingredient in the definition of (almost-)crystallographic groups.

Definition 3.2.24. Let G be a connected, simply connected, nilpotent Lie group. A *lattice* of G is a discrete, cocompact subgroup N of G. The compact quotient space $N\backslash G$ is called a *nilmanifold* and its fundamental group is exactly N.

Note that the dimension of the manifold $N\backslash G$ will coincide with the Hirsch length $h(N)$ of N.

Theorem 3.2.25 (see [Mal51; Rag72])**.**

(1) A lattice of a connected, simply connected, nilpotent Lie group must be a finitely generated, torsion-free, nilpotent group.

(2) Conversely, if N is a finitely generated, torsion-free, nilpotent group, then there exists a unique (up to isomorphism) connected, simply connected, nilpotent Lie group G such that N is a lattice of G. This G is called the Mal'cev completion *of N.*

(3) If φ is an endomorphism (automorphism) of a finitely generated, torsion-free, nilpotent group N, then φ extends uniquely to a Lie group endomorphism (automorphism) $\tilde{\varphi} : G \to G$ of the Mal'cev completion G of N.

3.2.3 Free Lie algebras and free nilpotent Lie algebras

In what follows, we will define a so-called Hall basis of a free (nilpotent) Lie algebra. More details (and a more formal treatment) can be found in e.g. [Ser92, Chapter IV].

Definition 3.2.26. The *free Lie algebra* $\mathfrak{f}_r$ is the Lie algebra generated by r elements $X_1, X_2, \ldots, X_r$, on whose Lie bracket we only impose the relations of alternativity and the Jacobi identity.

Note that this does not mean that r is the dimension of the vector space underlying $\mathfrak{f}_r$. For example, the element $[X_1, X_2]$ is not spanned by $X_1, X_2, \ldots, X_r$.

A Hall basis H of $\mathfrak{f}_r$ is a vector space basis of $\mathfrak{f}_r$ that is totally ordered, and which is constructed inductively as a union $H = \cup_{n\in\mathbb{N}} H_n$ with H_n consisting of n-fold Lie brackets, according to the following rules:

- $H_1 := \{X_1, X_2, \ldots, X_r\}$, and the order is given by $X_1 < X_2 < \cdots < X_r$.

We now proceed inductively: let $n \geq 2$ and assume that H_k has been defined for all $k < n$ and that $\cup_{k=1}^{n-1} H_k$ has been given a total order.

- We define H_n as the set of elements of the form $[Y, Z]$ with $Y \in H_k$, $Z \in H_l$ where
  - $k + l = n$,
  - $Y < Z$,
  - if $Z = [Z_1, Z_2]$ for some $Z_1 \in H_{l_1}$, $Z_2 \in H_{l_2}$, then $Z_1 \leq Y$.
- We extend the order on $\cup_{k=1}^{n-1} H_k$ to an order on $\cup_{k=1}^{n} H_k$ by choosing any total order on H_n, and setting $X < Y$ for all $X \in H_k, Y \in H_n$ with $k < n$.

Example 3.2.27. The elements of H_2 are of the form

$$[X_i, X_j], \quad \text{with } 1 \leq i < j \leq r.$$

The elements of H_3 are of the form

$$[X_i, [X_j, X_k]], \quad \text{with } 1 \leq j < k \leq r \text{ and } 1 \leq j \leq i \leq r.$$

The elements of H_4 depend on the choice of ordering we took for the elements of H_2.

In a way similar to how we defined free nilpotent groups as quotients of free groups, we can define free nilpotent Lie algebras.

Definition 3.2.28. Let $\mathfrak{f}_r$ be the free Lie algebra with r generators. For any $c \geq 1$, the quotient

$$\mathfrak{g}_{r,c} := \frac{\mathfrak{f}_r}{\gamma_{c+1}(\mathfrak{f}_r)}$$

is a Lie algebra of nilpotency class c called the *free nilpotent Lie algebra* of rank r and nilpotency class c.

Let $G_{r,c}$ be the Mal'cev completion of $N_{r,c}$, the free nilpotent group of rank r and nilpotency class c. Then the Lie algebra corresponding to $G_{r,c}$ is exactly the free nilpotent Lie algebra $\mathfrak{g}_{r,c}$. If $H = \cup_{n \in \mathbb{N}} H_n$ is a Hall basis of $\mathfrak{f}_r$, then the natural projections of the elements of length at most c (i.e. $H_1 \cup H_2 \cup \cdots \cup H_c$) form a basis of $\mathfrak{g}_{r,c}$, which we also call a Hall basis.

Proposition 3.2.29 (see [Wit37, Satz 3])**.** *Let $H = \cup_{k=1}^{c} H_k$ be a Hall basis of $\mathfrak{g}_{r,c}$ and let $k \leq c$. The dimension of $\gamma_k(\mathfrak{g}_{r,c})/\gamma_{k+1}(\mathfrak{g}_{r,c})$ is given by*

$$\# H_k = \frac{1}{k} \sum_{d \mid k} \mu(d) r^{k/d},$$

with μ the Möbius function from eq. (3.1).

3.3 Crystallographic groups

The class of almost-crystallographic groups is a natural generalisation of the class of crystallographic groups, hence, let us start with exploring the crystallographic groups. Let $\mathbb{R}^n$ be the Euclidean space, and denote the set of isometries on this space by $\mathrm{Isom}(\mathbb{R}^n)$. Any isometry can be seen as a map

$$\mathbb{R}^n \to \mathbb{R}^n : x \mapsto Ax + a,$$

with $A \in \mathrm{O}(n)$ and $a \in \mathbb{R}^n$. Thus, we can identify this isometry with an element $(a, A) \in \mathbb{R}^n \times \mathrm{O}(n)$. In particular, the composition of two elements (a, A), (b, B) is the map

$$\mathbb{R}^n \to \mathbb{R}^n : x \mapsto A(Bx + b) + a = ABx + Ab + a,$$

hence $(a, A)(b, B) = (Ab + a, AB)$, and thus $\mathrm{Isom}(\mathbb{R}^n)$ is actually the semidirect product group $\mathbb{R}^n \rtimes \mathrm{O}(n)$. Similarly, we can see that the affine group $\mathrm{Aff}(\mathbb{R}^n)$ is the semidirect product $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$, and clearly $\mathrm{Isom}(\mathbb{R}^n) \subseteq \mathrm{Aff}(\mathbb{R}^n)$. Note that the groups $\mathrm{Isom}(\mathbb{R}^n)$ and $\mathrm{Aff}(\mathbb{R}^n)$ are both Lie groups.

Definition 3.3.1. An n-dimensional *crystallographic group* is a discrete, cocompact subgroup of $\mathrm{Isom}(\mathbb{R}^n)$. A *Bieberbach group* is a torsion-free crystallographic group.

Example 3.3.2. We list some examples of crystallographic groups.

(1) We may identify $\mathbb{Z}^n$ with the subgroup $\mathbb{Z}^n \times \{\mathbb{1}_n\}$ of $\mathrm{Isom}(\mathbb{R}^n)$, which is an n-dimensional Bieberbach group.

(2) Consider the group generated by the isometries

$$a := \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

$$b := \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

This is a two-dimensional Bieberbach group, and can be presented by

$$\left\langle a, b \mid ab = ba^{-1} \right\rangle.$$

(3) The semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{-1, 1\}$ acts on $\mathbb{Z}$ by $(\pm 1) \cdot x = \pm x$, is isomorphic to the subgroup of $\mathrm{Isom}(\mathbb{R})$ generated by $(1, 1)$ and $(0, -1)$. This is a one-dimensional crystallographic group that is not torsion-free, and is isomorphic to the infinite dihedral group $\mathcal{D}_\infty$ and the free product $\mathbb{Z}_2 * \mathbb{Z}_2$.

3.3.1 Bieberbach theorems

The structure of crystallographic groups is described by the so-called Bieberbach theorems which were proved by Bieberbach and Frobenius [Bie11; Bie12; Fro11].

Theorem 3.3.3 (First Bieberbach theorem)**.** *Let* $\Gamma \subseteq \mathrm{Isom}(\mathbb{R}^n)$ *be an* n*-dimensional crystallographic group. Then the group of translations* $N := \Gamma \cap \mathbb{R}^n$ *is a lattice of* $\mathbb{R}^n$ *and has finite index in* $\mathbb{R}^n$*.*

Being a lattice in $\mathbb{R}^n$ implies being isomorphic to $\mathbb{Z}^n$. In particular, this means a crystallographic group fits in a short exact sequence

$$1 \longrightarrow \mathbb{Z}^n \xrightarrow{\ i\ } \Gamma \longrightarrow F \longrightarrow 1$$

with $i(\mathbb{Z}^n)$ maximal abelian in Γ and F finite. We call F the *holonomy group* of Γ. This short exact sequence induces a faithful representation $\rho : F \to \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$ called the *holonomy representation*, meaning that we can see F as a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$. Also note that $i(\mathbb{Z}^n)$ being maximal abelian implies it is a characteristic subgroup of Γ, though it is not necessarily a fully characteristic subgroup.

Example 3.3.4. Consider the crystallographic group Γ generated by

$$a := \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \quad b := \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \quad c := \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right).$$

The map φ defined by $\varphi(a) = c$ and $\varphi(b) = \varphi(c) = 1$ is an endomorphism of Γ that does not leave the translation subgroup invariant.

In [Zas48], Zassenhaus proved the following converse to the first Bieberbach theorem.

Theorem 3.3.5. *Let* G *be any group that fits in a short exact sequence as above, where* F *is finite and* $i(\mathbb{Z}^n)$ *is maximal abelian in* G*. Then there exists an embedding* $j : G \to \mathrm{Isom}(\mathbb{R}^n)$ *such that* $j(G)$ *is an* n*-dimensional crystallographic group.*

The second Bieberbach theorem describes the structure of isomorphisms between crystallographic groups.

Theorem 3.3.6 (Second Bieberbach theorem)**.** *Let* Γ, Γ' *be* n*-dimensional crystallographic groups and* $\varphi : \Gamma \to \Gamma'$ *be an isomorphism. Then there exists some* $\delta \in \mathrm{Aff}(\mathbb{R}^n)$ *such that*

$$\varphi(\gamma) = \delta\gamma\delta^{-1}$$

for all $\gamma \in \Gamma$*, i.e.* φ *is the restriction to* Γ *of some inner automorphism* ι_δ *of* $\mathrm{Aff}(\mathbb{R}^n)$*.*

Since the translation subgroup of a crystallographic group Γ is isomorphic to $\mathbb{Z}^n$ and the holonomy group is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{Z})$, there exists a finite set of vectors $a_1, a_2, \ldots, a_k \in \mathbb{R}^n$ and a finite set of matrices $A_1, A_2, \ldots, A_k \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$\Gamma \cong \langle (\mathbb{Z}^n, \mathbb{1}_n), (a_1, A_1), (a_2, A_2), \ldots, (a_k, A_k) \rangle,$$

$$F \cong \langle A_1, A_2, \ldots, A_k \rangle.$$

The group Γ' generated by $\mathbb{Z}^n$ and the set $\{(a_1, A_1), (a_2, A_2), \ldots, (a_k, A_k)\}$ is isomorphic to Γ, but may no longer be a subgroup of $\mathrm{Isom}(\mathbb{R}^n)$. We will also call such $\Gamma' \subseteq \mathrm{Aff}(\mathbb{R}^n)$ with translation subgroup $\mathbb{Z}^n$ a crystallographic group, and remark that Γ' is conjugate to Γ inside $\mathrm{Aff}(\mathbb{R}^n)$.

While we lose the geometric aspect of being a group of isometries, this allows us to restate the second Bieberbach theorem in the following (helpful) way.

Proposition 3.3.7. *Let* Γ *be an* n*-dimensional crystallographic group whose translation subgroup is exactly* $\mathbb{Z}^n$ *and whose holonomy group* F *is a subgroup of* $\mathrm{GL}_n(\mathbb{Z})$*. Let* $\varphi \in \mathrm{Aut}(\Gamma)$ *be an automorphism. Then there exist some* $d \in \mathbb{R}^n$*,* $D \in N_{\mathrm{GL}_n(\mathbb{Z})}(F)$ *such that*

$$\varphi(\gamma) = (d, D)\gamma(d, D)^{-1}$$

for all $\gamma \in \Gamma$*.*

Proof. Γ is conjugate (in $\mathrm{Aff}(\mathbb{R}^n)$) to a crystallographic group $\Gamma' \subseteq \mathrm{Isom}(\mathbb{R}^n)$, hence from the second Bieberbach theorem, we know that there exist $d \in \mathbb{R}^n$, $D \in \mathrm{GL}_n(\mathbb{R})$ such that $\varphi(\gamma) = (d, D)\gamma(d, D)^{-1}$ for all $\gamma \in \Gamma$. Because $\mathbb{Z}^n$ is a characteristic subgroup of Γ, φ induces an automorphism $\varphi' : F \to F : A \mapsto DAD^{-1}$, and therefore $D \in N_{\mathrm{GL}_n(\mathbb{Z})}(F)$. □

This normaliser $N_{\mathrm{GL}_n(\mathbb{Z})}(F)$ gives us information about the (in)finiteness of the outer automorphism group $\mathrm{Out}(\Gamma)$.

Theorem 3.3.8 (see [Szc12, Section 5.1])**.** *Let* Γ *be an* n*-dimensional crystallographic group whose translation subgroup is exactly* $\mathbb{Z}^n$ *and whose holonomy group* F *is a subgroup of* $\mathrm{GL}_n(\mathbb{Z})$*. Then* $\# N_{\mathrm{GL}_n(\mathbb{Z})}(F) = \infty$ *if and only if* $\# \mathrm{Out}(\Gamma) = \infty$*.*

Let us now agree on some notation. If Γ is a crystallographic group with holonomy group $F \subseteq \mathrm{GL}_n(\mathbb{Z})$, and $\varphi \in \mathrm{Aut}(\Gamma)$ is conjugation by $(d, D) \in \mathrm{Aff}(\mathbb{R}^n)$, we set:

$$N_F := N_{\mathrm{GL}_n(\mathbb{Z})}(F),$$

$$\xi_{(d,D)} := \Gamma \to \Gamma : \gamma \mapsto (d, D)\gamma(d, D)^{-1}.$$

The third Bieberbach theorem talks about the finiteness of the number of crystallographic groups.

Theorem 3.3.9 (Third Bieberbach theorem)**.** *For any $n \in \mathbb{N}$, there are (up to isomorphism) only finitely many n-dimensional crystallographic groups.*

The crystallographic groups have been classified up to dimension 6. In table 3.1 we give the number of crystallographic groups and Bieberbach groups in every (known) dimension. We also mention the number of crystallographic groups with finite outer automorphism group, as this property will be crucial later in this thesis.

dim	# cryst. groups	# with $\#N_F < \infty$	# Bieberbach groups
1	2	2	1
2	17	15	2
3	219	204	10
4	4 783	4 388	74
5	222 018	204 768	1 060
6	28 927 915	26 975 265	38 746

Table 3.1: Number of crystallographic groups

The 2-dimensional groups were classified by Fedorov and Pólya [Fed91; Pól24], the 3-dimensional groups by Barlow, Fedorov and Schönflies [Bar94; Fed91; Sch91], the 4-dimensional groups by Brown, Bülow, Neubüser, Wondratscheck and Zassenhaus [Bro+78] and the 5- and 6-dimensional groups by Plesken and Schulz [PS00], who made use of CARAT [Car06].

3.3.2 Flat manifolds and orbifolds

There is a geometrical interpretation of crystallographic groups.

Proposition 3.3.10. *Let Γ be an n-dimensional crystallographic group. Then Γ (as a subgroup of $\mathrm{Aff}(\mathbb{R}^n)$) acts on $\mathbb{R}^n$, and this action has the following properties:*

- Γ *acts properly discontinuously on* $\mathbb{R}^n$*, i.e.*

$$\#\{\gamma \in \Gamma \mid \gamma \cdot K \cap K \neq \varnothing\} < \infty$$

  for any compact subset $K \subseteq \mathbb{R}^n$*.*

- Γ *acts cocompactly on* $\mathbb{R}^n$*, i.e. the orbit space* $\Gamma\backslash\mathbb{R}^n$ *is compact.*

- *The action of* Γ *on* $\mathbb{R}^n$ *is free if and only if* Γ *is torsion-free.*

In particular, this means that $\Gamma\backslash\mathbb{R}^n$ is a compact topological space, and the manifold structure from $\mathbb{R}^n$ induces a manifold (orbifold) structure on $\Gamma\backslash\mathbb{R}^n$ if Γ is a Bieberbach (crystallographic) group, and the (orbifold) fundamental group is exactly Γ.

Remark 3.3.11. Note that it is important that we talk about the orbifold fundamental group and not the usual topological fundamental group when Γ is not torsion-free. It can be proven that the topological fundamental group of $\Gamma\backslash\mathbb{R}^n$ is exactly $\Gamma/\langle\tau(\Gamma)\rangle$, whereas the orbifold fundamental group is Γ.

Moreover, because Γ (as a subgroup of $\text{Isom}(\mathbb{R}^n)$) acts on $\mathbb{R}^n$ by isometries, the (flat) Riemannian metric on $\mathbb{R}^n$ induces a metric on $\Gamma\backslash\mathbb{R}^n$. Thus, $\Gamma\backslash\mathbb{R}^n$ is a compact, flat manifold (orbifold). The converse is also true: any flat manifold (orbifold) can be obtained as a quotient $\Gamma\backslash\mathbb{R}^n$ with Γ a Bieberbach (crystallographic) group.

Example 3.3.12. Consider the crystallographic groups from example 3.3.2.

(1) $\mathbb{Z}^n\backslash\mathbb{R}^n$ is the n-dimensional (flat) torus, the direct product of n copies of the circle S^1.

(2) If $\Gamma \cong \langle a, b \mid ab = ba^{-1}\rangle$, then $\Gamma\backslash\mathbb{R}^2$ is the Klein bottle.

(3) If $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}_2$, then $\Gamma\backslash\mathbb{R}$ is a closed interval.

This allows us to interpret the second and third Bieberbach theorems geometrically. The second Bieberbach theorem states that, up to affine equivalence, a flat manifold (orbifold) is completely determined by its (orbifold) fundamental group. The third Bieberbach theorem states that for any dimension $n \in \mathbb{N}$, there are only finitely many compact, flat manifolds (orbifolds).

3.3.3 Generalised Hantzsche-Wendt groups and manifolds

A subclass of the Bieberbach groups that has received special attention, is the class of (generalised) Hantzsche-Wendt groups, see for example [DDM04; DHS09; DP09; MR99b; RS05].

Definition 3.3.13. An n-dimensional Bieberbach group Γ with holonomy group isomorphic to $\mathbb{Z}_2^{n-1}$ is called a *generalised Hantzsche-Wendt group*, or GHW group. If Γ is *orientable*, i.e. $\det(A) = 1$ for every $A \in F$, it is called a *Hantzsche-Wendt group*, or HW group.

The corresponding flat manifolds are generalised Hantzsche-Wendt manifolds, or Hantzsche-Wendt manifolds if they are orientable.

Example 3.3.14. The (classical) Hantzsche-Wendt group is the Bieberbach group generated by the isometries

$$\left(\begin{pmatrix}\frac{1}{2}\\0\\0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\right), \qquad \left(\begin{pmatrix}0\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\right).$$

The corresponding flat manifold is called the Hantzsche-Wendt manifold [HW35]. In fact, it is the only Hantzsche-Wendt group of dimension 3.

Theorem 3.3.15 (See [RS05, Section 2]). *A Hantzsche-Wendt group is necessarily of odd dimension.*

We need the following theorem to provide a nice presentation of a GHW group.

Theorem 3.3.16 (see [RS05, Theorem 3.1]). *Let Γ be an n-dimensional GHW group. Then there exists a presentation of Γ such that for every $A \in F$, A is a diagonal matrix.*

Definition 3.3.17. We say that a HW group is in *standard form* if it is generated by $\mathbb{Z}^n$ and $(a_1, A_1), \ldots, (a_n, A_n)$ where A_i is the diagonal matrix with 1 on the i-th place and -1 on the other places, and where $a_i \in \{0, 1/2\}^n$.

Every HW group is isomorphic to a HW group in standard form, hence from now on we will always assume that a HW group is in standard form. Let Γ be a Hantzsche-Wendt group with standard generators $(a_1, A_1), \ldots, (a_n, A_n)$. We define the $n \times n$-matrix A as

$$A := (a_{ij})_{ij},$$

where a_{ij} is the $j-$th coordinate of a_i. This is called the matrix associated to a HW group.

Example 3.3.18. The Hantzsche-Wendt group with the generators given in example 3.3.14 is in standard form with associated matrix

$$\begin{pmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{2}\end{pmatrix}.$$

One can then consider the problem of which matrices with entries in $\{0, 1/2\}$ are associated matrices of HW groups. The following proposition provides a necessary and sufficient condition for this.

Proposition 3.3.19 (see [MR99b, Proposition 1.2])**.** *Consider the crystallographic group $\Gamma = \langle \mathbb{Z}^n, (a_1, A_1), \dots, (a_n, A_n)\rangle$, with n odd, A_i as before and $a_i \in \{0, 1/2\}^n$. Then Γ is a HW group if and only if, for any $I \subsetneq \{1, 2, \dots, n\}$ with $\#I$ odd we have:*

$$\exists j \in I : \#\{i \in I \mid a_{ij} = 1/2\} \text{ is odd.}$$

In particular, for each fixed j, we have that $a_{jj} = 1/2$ and $\#\{i \mid a_{ij} = 1/2\}$ is even.

This was used by Miatello en Rossetti to classify the Hantzsche-Wendt groups up to dimension 7 in [MR99a].

3.4 Almost-crystallographic groups

In this section, we generalise the previous section by going from abelian groups to nilpotent groups. Let G be a connected, simply connected, nilpotent Lie group. We define $\mathrm{Aff}(G)$ as the semidirect product $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ where multiplication is defined by $(d_1, D_1)(d_2, D_2) = (d_1 D_1(d_2), D_1 \circ D_2)$. Then $\mathrm{Aff}(G)$ acts on G by

$$(d, D)(g) = dD(g) \text{ for all } (d, D) \in \mathrm{Aff}(G) \text{ and all } g \in G.$$

We will often consider G as a subgroup of $\mathrm{Aff}(G)$ by identifying $g \in G$ with (g, id_G). Let C be a maximal compact subgroup of $\mathrm{Aut}(G)$, then $G \rtimes C$ is a subgroup of $\mathrm{Aff}(G)$. Such C is unique up to inner conjugation in $\mathrm{Aut}(G)$. Note that the group $G \rtimes C$ can be interpreted as a group of isometries of G, see [Dek18, Section 3].

Definition 3.4.1. An n-dimensional almost-crystallographic group modelled on the Lie group G is a discrete, cocompact subgroup of $G \rtimes C$. The dimension of Γ is defined as the dimension of G. An *almost-Bieberbach* group is a torsion-free almost-crystallographic group.

Example 3.4.2. Consider the following examples of almost-crystallographic groups:

(1) Consider the family of groups N_k defined as

$$N_k := \left\{ \begin{pmatrix} 1 & x & \frac{y}{k} \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\},$$

with $k \in \mathbb{N}$. Every group in this family is a lattice of the Heisenberg group $H_3(\mathbb{R})$. Note that N_k is actually isomorphic to the group from example 3.1.10.

(2) Let $k \in 2\mathbb{N}$ and define the automorphism $\varphi : H_3(\mathbb{R}) \to H_3(\mathbb{R})$ as follows:

$$\varphi(\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}) := \begin{pmatrix} 1 & -z & y - \frac{xz}{2} \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Fix some $k \in \mathbb{N}$, and define Γ as

$$\Gamma := \langle (N_k, \mathrm{id}), (\begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \varphi) \rangle.$$

This is an almost-Bieberbach group. If we replace the $\frac{1}{4}$ by 0, we get an almost-crystallographic group which is not torsion-free.

3.4.1 Generalised Bieberbach theorems

The three Bieberbach theorems have been generalised to almost-crystallographic groups.

Theorem 3.4.3 (Generalised first Bieberbach theorem, see [Aus60])**.** *Let Γ be an almost-crystallographic group modelled on the Lie group G. Then the group of translations $N := \Gamma \cap G$ is a lattice in G and has finite index in G.*

Similar to the crystallographic case, being a lattice implies that the translation group is a finitely generated, torsion-free, nilpotent group. Thus, an almost-crystallographic group fits in a short exact sequence

$$1 \longrightarrow N \xrightarrow{\ i\ } \Gamma \longrightarrow F \longrightarrow 1$$

with N a finitely generated, torsion-free, nilpotent group, $i(N)$ maximal nilpotent in Γ and F finite. Once again, we call F the *holonomy group* of Γ and $i(N)$ is a characteristic subgroup of Γ. Due to the bijectivity of the log-map, there is a one-to-one relation between $\mathrm{Aut}(G)$ and Lie algebra automorphisms of $\mathfrak{g}$, the Lie algebra associated with G. Hence, by fixing a basis for $\mathfrak{g}$, there exists a faithful representation $\rho : F \to \mathrm{GL}_n(\mathbb{R})$.

Again, $i(N)$ is not necessarily fully characteristic, but it does contain a fully characteristic subgroup of finite index.

Lemma 3.4.4 (see [LL06, Lemma 3.1]). *Let Γ be an almost-crystallographic group with translation subgroup N. Then Γ contains a fully characteristic, finite index subgroup $H \subseteq N$.*

The converse to the generalised first Bieberbach theorem holds as well.

Theorem 3.4.5 (see [Lee88]). *Let Γ be any group that fits in a short exact sequence as above, where F is finite, N is finitely generated, torsion-free and nilpotent, and $i(N)$ is maximal nilpotent in Γ. Then Γ is (isomorphic to) an almost-crystallographic group.*

Theorem 3.4.6 (Generalised second Bieberbach theorem, see [LR85]). *Let Γ, Γ' be n-dimensional almost-crystallographic groups modelled on a Lie group G and $\varphi : \Gamma \to \Gamma'$ be an isomorphism. Then there exists some $\delta \in \operatorname{Aff}(G)$ such that*

$$\varphi(\gamma) = \delta\gamma\delta^{-1}$$

for all $\gamma \in \Gamma$, i.e. φ is the restriction to Γ of some inner automorphism ι_δ of $\operatorname{Aff}(G)$.

We will again use ξ_δ to denote an automorphism that is conjugation by $\delta \in \operatorname{Aff}(G)$.

Generalising the third Bieberbach theorem is more tricky: for any dimension $n \geq 3$, there are infinitely many (non-isomorphic) almost-crystallographic groups. This is true even if we only consider the almost-crystallographic groups modelled on a fixed connected, simply connected, nilpotent Lie group. For example, example 3.4.2(1) gives an infinite family of almost-crystallographic groups modelled on the Heisenberg group $H_3(\mathbb{R})$.

From an algebraic point of view, however, the translation subgroup of a crystallographic group is always isomorphic to $\mathbb{Z}^n$. Hence, we can reformulate the third Bieberbach theorem as saying that for any dimension n, there are only finitely many crystallographic groups with translation subgroup (isomorphic to) $\mathbb{Z}^n$. This statement can then be generalised to the almost-crystallographic case.

Theorem 3.4.7 (Generalised third Bieberbach theorem, see [DIM94; Lee88]). *Let N be a finitely generated, torsion-free, nilpotent group. There are (up to isomorphism) only finitely many almost-crystallographic groups for which the translation subgroup is isomorphic to N.*

The 3-dimensional almost-crystallographic groups were fully classified by Dekimpe in [Dek96]. This book also contains a partial classification of the 4-dimensional almost-crystallographic groups, including a complete classification of the 4-dimensional almost-Bieberbach groups.

The following lemma is at the basis of this classification:

Lemma 3.4.8 (see [Dek96, Lemma 2.4.2]). *Let* Γ *be an almost-crystallographic group with translation subgroup* N *of nilpotency class* c*, and define* $Z := \sqrt[N]{\gamma_c(N)}$*. Then* Γ/Z *is an almost-crystallographic group with translation subgroup* N/Z *of nilpotency class* $c-1$*.*

Using lemma 3.1.6(iv), we may upgrade this lemma.

Corollary 3.4.9 (see [Dek96, Lemma 2.4.2]). *Let* Γ *be an almost-crystallographic group with translation subgroup* N *of nilpotency class* c*, and define* $Z := \sqrt[N]{\gamma_k(N)}$ *with* $k \leq c$*. Then* Γ/Z *is an almost-crystallographic group with translation subgroup* N/Z *of nilpotency class* $k-1$*. In particular, if* $k = 2$*, then* Γ/Z *is crystallographic.*

This means that every almost-crystallographic group has a quotient group that is crystallographic. We may thus classify the almost-crystallographic groups into families based on the nilpotency class of their translation subgroups and on this crystallographic quotient.

3.4.2 Infra-nilmanifolds and orbifolds

The entirety of proposition 3.3.10 generalises to the case of almost-crystallographic groups. Hence, $\Gamma\backslash G$ is again a compact manifold (orbifold) if Γ is an almost-Bieberbach (almost-crystallographic) group modelled on the Lie group G. We call such manifold (orbifold) an *infra-nilmanifold* (*infra-nilorbifold*).

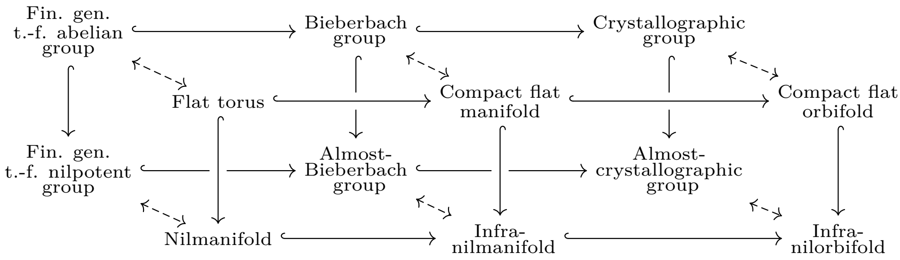
A nice result by Gromov and Ruh shows that infra-nilmanifolds are generalisations of flat manifolds.

Theorem 3.4.10 (see [Gro78; Ruh82]). *A compact manifold* M *is infra-nil if and only if it is almost flat, i.e. for any* $\varepsilon > 0$*, there exists a Riemannian metric* g_ε *such that*

- $\operatorname{diam}(M, g_\varepsilon) \leq 1$,
- $|K_{g_\varepsilon}| < \varepsilon$ *with* K_{g_ε} *the sectional curvature.*

This result has been generalised to infra-nilorbifolds and almost flat orbifolds by Ghanaat [Gha97].

The following diagram summarises the underlying relations between most of the concepts introduced in this chapter.



3.4.3 Self-maps on infra-nilmanifolds

Define the semigroup

$$\operatorname{aff}(G) := G \rtimes \operatorname{End}(G).$$

The following theorem generalises the generalised second Bieberbach theorem even further, namely from automorphisms to endomorphisms.

Theorem 3.4.11 (see [Lee95, Theorem 1.1])**.** *Let Γ be an n-dimensional almost-crystallographic group modelled on a Lie group G and $\varphi : \Gamma \to \Gamma$ be an endomorphism. Then there exists some $(d, D) \in \operatorname{aff}(G)$ such that*

$$\varphi(\gamma) \circ (d, D) = (d, D) \circ \gamma$$

for all $\gamma \in \Gamma$.

In fact, this can even be generalised to any morphism between two almost-crystallographic groups, not necessarily modelled on the same Lie group, see [Dek18, Theorem 5.1].

This theorem allows us to construct a self-map on an infra-nilmanifold induced by an endomorphism.

Lemma 3.4.12. *Let Γ be an almost-Bieberbach group modelled on a Lie group G, and let $\varphi = \xi_{(d,D)}$ be an endomorphism on Γ. Then*

$$f_\varphi := \overline{(d, D)} : \Gamma\backslash G \to \Gamma\backslash G : \Gamma \cdot g \mapsto \Gamma \cdot dD(g)$$

is a well-defined map.

Remark 3.4.13. If we take (d, D) as the reference lift of $f_\varphi = \overline{(d, D)}$, then the equation $\varphi(\gamma) \circ (d, D) = (d, D) \circ \gamma$ means that the induced endomorphism on the fundamental group of $\Gamma\backslash G$ is exactly φ, i.e. $f_{\varphi *} = \varphi$. Picking a different lift will change the induced morphism by an inner automorphism.

Theorem 3.4.14. *Let f, g be two self-maps on an infra-nilmanifold $\Gamma\backslash G$. If the induced endomorphisms f_*, g_* on Γ (with respect to certain lifts) are equal up to inner automorphism, then f and g are homotopic.*

Corollary 3.4.15. *Let $f : \Gamma\backslash G \to \Gamma\backslash G$ be a self-map on an infra-nilmanifold. Then there exists an affine map $(d, D) \in \mathrm{aff}(G)$ such that the induced map $\overline{(d, D)}$ on $\Gamma\backslash G$ is homotopic to f. We call (d, D) an* affine homotopy lift *of f.*

In general, the affine map (d, D) is far from unique. The Lie group endomorphism D is determined up to an inner automorphism of G, and the translation part d is determined up to an element of $\mathrm{Fix}(F) \subseteq G$, the elements that are invariant under the action of the holonomy group F on G [Lee95, Proposition 1.4]. A first remark we can make, is that if Γ is crystallographic, then D is unique. A second remark is that if we consider a self-map on a nilmanifold, then we may always pick $d = 1_G$ for the affine homotopy lift. This leads us to the following theorem.

Theorem 3.4.16 (see [McC97, Lemma 2.7])**.** *Let $f : N\backslash G \to N\backslash G$ be a self-map on a nilmanifold. Then there exists a Lie group endomorphism $D : G \to G$ such that D is an affine homotopy lift of f.*

The situation for infra-nilorbifolds is more difficult, since many (often non-equivalent) notions of an orbifold map exist. For all common notions of orbifold maps, it is possible to generalise lemma 3.4.12: any endomorphism of the orbifold fundamental group will induce an orbifold map. However, the converse need not be true: an orbifold map does not necessarily admit a “global” lift to the universal orbifold cover.

Chapter 4

Reidemeister-Nielsen fixed point theory on almost-crystallographic groups

Having introduced both Reidemeister-Nielsen fixed point theory and almost-crystallographic groups, we are now ready to combine the two. This will result in algebraic formulas to compute Lefschetz, Nielsen and Reidemeister numbers for infra-nilmanifolds, and some results on Reidemeister numbers of almost-crystallographic groups.

4.1 Nilmanifolds

Theorem 4.1.1 (Anosov theorem, see [Ano85; FH86])**.** *Let $f : N \to N$ be a self-map on a nilmanifold. Then $N(f) = |L(f)|$.*

The idea behind this theorem is that every fixed point class of a self-map f on a nilmanifold has the same index, and this index is always -1, 0 or 1. In general, we say that the *Anosov relation* holds for a self-map f whenever $N(f) = |L(f)|$.

There exist easy formulas for the Nielsen, Lefschetz and Reidemeister number of self-maps on nilmanifolds.

Theorem 4.1.2 (see [Ano85]). *Let $f : M \to M$ be a self-map on a nilmanifold $M = N\backslash G$. Let $D : G \to G$ be an affine homotopy lift of f, then*

$$L(f) = \det(\mathbb{1} - D_*),$$

$$N(f) = |\det(\mathbb{1} - D_*)|,$$

$$R(f) = |\det(\mathbb{1} - D_*)|_\infty.$$

Theorem 4.1.3 (see [HK97]). *Let $f : N \to N$ be a self-map on a nilmanifold. Then either all fixed point classes are essential, or all of them are inessential.*

In the literature, a space for which this property holds is called *weakly Jiang*.

Corollary 4.1.4. *Let $f : N \to N$ be a self-map on a nilmanifold. Then $R(f) = |N(f)|_\infty$.*

Due to the connection between nilmanifolds and finitely generated, torsion-free, nilpotent groups, theorem 4.1.2 also implies the following:

Theorem 4.1.5. *Let N be a finitely generated, torsion-free, nilpotent group, and $\varphi \in \mathrm{End}(N)$. Let G be the Mal'cev completion of N and let $D : G \to G$ be an affine homotopy lift of φ. Denote by D_* the induced endomorphism on $\mathfrak{g}$. Then*

$$R(\varphi) = |\det(\mathbb{1} - D_*)|_\infty.$$

Theorem 4.1.6 (see [Rom11, Theorem 2.6]). *Let N be a finitely generated, nilpotent group. Let*

$$N = N_1 \geq N_2 \geq \cdots \geq N_c \geq N_{c+1} = 1$$

be a central series of N, such that all factors N_k/N_{k+1} are torsion-free. If $\varphi \in \mathrm{End}(N)$ and $\varphi(N_k) \subseteq N_k$ for all k, then

$$R(\varphi) = \prod_{k=1}^{c} R((\varphi)_k),$$

where $(\varphi)_k$ is the induced endomorphism on the factor N_k/N_{k+1}.

Proof. We prove this by induction on the length c of the central series. If $c = 1$, the result follows trivially. Let $c > 1$ and assume the theorem holds for a central series of length $c - 1$. Let $\varphi \in \mathrm{End}(N)$, then $\varphi(N_c) \subseteq N_c$ and hence we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_c & \xrightarrow{i} & N & \xrightarrow{p} & N/N_c & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle (\varphi)_c} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} & & \\ 1 & \longrightarrow & N_c & \xrightarrow{i} & N & \xrightarrow{p} & N/N_c & \longrightarrow & 1 \end{array}$$

The quotient N/N_c is a finitely generated, nilpotent group with central series

$$N_1/N_c \geq N_2/N_c \geq \cdots \geq N_{c-1}/N_c \geq 1$$

of length $c-1$. Every factor of this series is of the form $(N_k/N_c)/(N_{k+1}/N_c)$, which is isomorphic to N_k/N_{k+1} by the third isomorphism theorem, hence it is also torsion-free. Moreover, because of this natural isomorphism we know that for every induced automorphism $(\varphi')_k$ on $(N_k/N_c)/(N_{k+1}/N_c)$ it holds that

$$R((\varphi')_k) = R((\varphi)_k).$$

First, assume that $R(\varphi') = \infty$, in which case we find $R(\varphi) = \infty$ by lemma 2.5.10(1). Then by the induction hypothesis $R((\varphi)_k) = R((\varphi')_k) = \infty$ for some $k \in \{1, \ldots, c-1\}$, so the theorem holds in this case.

Next, suppose that $R(\varphi') < \infty$ and $R(\varphi_c) = \infty$. We then know from the induction hypothesis that $R((\varphi')_k) < \infty$ for every $k \in \{1, \ldots, c-1\}$. Note that $(N_k/N_c)/(N_{k+1}/N_c)$ is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$, so $(\varphi')_k$ can be seen as a matrix in $\mathrm{GL}_n(\mathbb{Z})$ and hence

$$R((\varphi')_k) = |\det(\mathbb{1} - (\varphi')_k)|_\infty < \infty,$$

or equivalently $\mathrm{Fix}((\varphi')_k) = \{1\}$. Suppose that $|\mathrm{Fix}(\varphi')| = \infty$. Because N/N_c is a finitely generated, torsion-free, nilpotent group, there exists some $gN_c \in N/N_c$ of infinite order such that

$$\varphi'(gN_c) = gN_c.$$

Suppose that $gN_c \in N_k/N_c$ but $gN_c \notin N_{k+1}/N_c$. Then

$$(\varphi')_k(gN_{k+1}/N_c) = gN_{k+1}/N_c,$$

which contradicts the fact that $\mathrm{Fix}((\varphi')_k) = \{1\}$ for all k. Thus $|\mathrm{Fix}(\varphi')| < \infty$ and by lemma 2.5.10(2) $R(\varphi) = \infty$. Again, the theorem holds in this case.

The final case is the one where $R(\varphi') < \infty$ and $R((\varphi)_c) < \infty$. Let $[g_1N_c]_{\varphi'}, \ldots,$ $[g_nN_c]_{\varphi'}$ represent the Reidemeister classes of φ' and $[c_1]_{(\varphi)_c}, \ldots, [c_m]_{(\varphi)_c}$ the Reidemeister classes of $(\varphi)_c$. Since $N_c \subseteq Z(N)$, we can apply lemma 2.5.10(3), hence it suffices to prove that every class $[c_ig_j]_\varphi$ represents a different Reidemeister class of φ to obtain

$$R(\varphi) = R(\varphi_c)R(\varphi'),$$

and then the theorem follows from the induction hypothesis.

Suppose there exists some $h \in N$ such that

$$c_i g_j = h c_a g_b \varphi(h)^{-1}.$$

Then by taking the projection to N/N_c we find

$$g_j N_c = p(c_i g_j) = p(h c_a g_b \varphi(h)^{-1}) = (hN_c)(g_b N_c)\varphi'(hN_c)^{-1}$$

hence $[g_j N_c]_{\varphi'} = [g_b N_c]_{\varphi'}$. Now assume that

$$c_i g_j = h c_a g_j \varphi(h)^{-1}.$$

If $h \in N_c \subseteq Z(N)$, then $[c_i]_{(\varphi)_c} = [c_a]_{(\varphi)_c}$, so assume $h \notin N_c$ and let N_k be the smallest group in the central series that contains h. Then

$$c_i c_a^{-1} N_{k+1} = g_j^{-1} h g_j \varphi(h)^{-1} N_{k+1} = [g_j, h^{-1}](h\varphi(h)^{-1})N_{k+1}.$$

As $c_i c_a^{-1} \in N_c \subseteq N_{k+1}$ and $[g_j, h^{-1}] \in N_{k+1}$, we find that

$$(\varphi)_k(hN_{k+1}) = hN_{k+1}.$$

This would mean that $\mathrm{Fix}((\varphi')_k) \neq 1$, which in turn would imply that $R(\varphi') = \infty$, which is a contradiction. Hence the result follows. □

We may also take a quick look at the zeta functions.

Theorem 4.1.7 (see [Fel00, Theorem 45]). *Let f be a self-map on a nilmanifold. Then the Nielsen zeta function $N_f(z)$ is rational. If the Reidemeister zeta function $R_f(z)$ exists, it equals the Nielsen zeta function (and is hence rational).*

One can simply mimic example 2.6.7 to prove this, with the addition of the case where $\lambda = 1$ for an eigenvalue λ of D_*.

4.2 Infra-nilmanifolds and almost-crystallographic groups

Just like for nilmanifolds, there exist algebraic formulas to calculate the Lefschetz, Nielsen and Reidemeister numbers of a self-map on an infra-nilmanifold.

Theorem 4.2.1 (see [KLL05; LL09] and [HLP12, Theorem 6.11])**.** *Let $\Gamma\backslash G$ be an infra-nilmanifold and let $F \subseteq \mathrm{Aut}(G)$ be the holonomy group of Γ. If f is a self-map on $\Gamma\backslash G$ with affine homotopy lift $(d, D) : G \to G$, then*

$$L(f) = \frac{1}{\#F} \sum_{A \in F} \det(\mathbb{1} - A_* D_*),$$

$$N(f) = \frac{1}{\#F} \sum_{A \in F} |\det(\mathbb{1} - A_* D_*)|,$$

$$R(f) = \frac{1}{\#F} \sum_{A \in F} |\det(\mathbb{1} - A_* D_*)|_\infty.$$

As one can expect from these formulas, for infra-nilmanifolds it is not true in general that $N(f) = |L(f)|$ and $R(f) = |N(f)|_\infty$.

Example 4.2.2. We provide a counterexample for each statement.

(1) Consider the group introduced in example 3.3.2(2), which was the fundamental group of the Klein bottle. Let f be a self-map on the Klein bottle inducing the automorphism $f_* = \xi_{(0,D)}$ with

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then using the formulas from theorem 4.2.1, we find that $N(f) = 2$ but $R(f) = \infty$.

(2) Consider the Bieberbach group generated by $\mathbb{Z}^n$ and

$$(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}).$$

Let f be a self-map inducing the endomorphism $f_* = \xi_{(0,D)}$ with

$$(0, \begin{pmatrix} 5 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}).$$

Then using the formulas from theorem 4.2.1, we find that $N(f) = 12$ but $L(f) = 0$.

However, the Nielsen and Reidemeister numbers are still closely related. The following easily follows from the averaging formulas.

Theorem 4.2.3. *Let f be a self-map on an infra-nilmanifold. Then:*

- $|L(f)| \leq N(f)$,
- *If $R(f) < \infty$, then $N(f) = R(f)$.*

Proposition 4.2.4 (see [FL15, Remark 9.4])**.** *Let $M = \Gamma\backslash G$ be an infra-nilmanifold and $\varphi = \xi_{(d,D)}$ an endomorphism of Γ inducing a self-map $f_\varphi = \overline{(d, D)} : M \to M$. If $N(f_\varphi) \neq 0$, then $N(f_\varphi) = \#\operatorname{Fix}(f_\varphi)$.*

The above proposition says that when $N(f_\varphi) \neq 0$, every map homotopic to f_φ has at least as many fixed points as the map f_φ.

There is also an easy criterion to determine whether a map has finite Reidemeister number. This criterion can be stated purely on the level of the almost-crystallographic group, hence we will talk about the Reidemeister number $R(\varphi)$ instead of $R(f_\varphi)$. In fact, this is corollary 2.5.15 adapted to the case of almost-crystallographic groups.

Theorem 4.2.5 (see [DP11, Corollary 3.12])**.** *Let Γ be an almost-crystallographic group with holonomy group F. Let $\varphi = \xi_{(d,D)}$ be an automorphism of Γ. Then*

$$R(\varphi) = \infty \iff \exists A \in F \text{ such that } \det(\mathbb{1} - A_* D_*) = 0.$$

The averaging formula for the Reidemeister number of a self-map given in theorem 4.2.1 can naturally be restated for endomorphisms, which we do below. The original proof of this formula was done in a topological way, though it is possible to give a purely group-theoretic proof.

Theorem 4.2.6 (averaging formula, see [HLP12, Theorem 6.11])**.** *Let Γ be an almost-Bieberbach group modelled on the Lie group G and with holonomy group F, and let $\varphi = \xi_{(d,D)}$ be an endomorphism of Γ. Then*

$$R(\varphi) = \frac{1}{\#F} \sum_{A \in F} |\det(\mathbb{1} - A_* D_*)|_\infty.$$

Proof. Let $H \subseteq N$ be a fully invariant, finite index subgroup of Γ, which exists due to lemma 3.4.4. H is, just like N, a finitely generated, torsion-free, nilpotent group. For any $\alpha = (a, A) \in \Gamma$, the map $\iota_\alpha \varphi|_H : H \to H$ is a well-defined endomorphism with $\iota_{aA(d)} AD : G \to G$ as affine homotopy lift. One can verify that

$$\operatorname{Fix}(\iota_\alpha \varphi|_H) \neq 1 \iff \det(\mathbb{1} - \iota_{aA(d)*} A_* D_*) = 0 \iff R(\iota_\alpha \varphi|_H) = \infty.$$

However, the inner automorphism $\iota_{aA(d)}$ has little impact here, since

$$\det(\mathbb{1} - \iota_{aA(d)*} A_* D_*) = \det(\mathbb{1} - A_* D_*).$$

If $R(\iota_\alpha \varphi|_H) = \infty$, then also $R(\varphi) = \infty$ by lemma 2.5.10(2) and hence the formula holds in this case. So now assume that $\mathrm{Fix}(\iota_\alpha \varphi|_H) = 1$ for all $\alpha \in \Gamma$.

Consider the short exact sequence of finite groups

$$1 \longrightarrow N/H \xrightarrow{\ i\ } \Gamma/H \xrightarrow{\ p\ } F \longrightarrow 1.$$

Fix a preimage $p^{-1}(A)$ for any $A \in F$. There is a one-to-one correspondence between the elements $\alpha H \in \Gamma/H$ and the products $i(aH)p^{-1}(A)$ with $aH \in N/H$ and $A \in F$. By proposition 2.5.16 and theorem 4.1.5, we have:

$$\begin{aligned}
R(\varphi) &= \frac{1}{[\Gamma : H]} \sum_{\alpha H \in \Gamma/H} R(\iota_\alpha \varphi|_H) \\
&= \frac{1}{[\Gamma : N][N : H]} \sum_{aH \in N/H} \sum_{A \in F} R(\iota_{i(aH)p^{-1}(A)} \varphi|_H) \\
&= \frac{1}{[\Gamma : N]} \frac{1}{[N : H]} \sum_{aH \in N/H} \sum_{A \in F} |\det(\mathbb{1} - A_* D_*)|_\infty \\
&= \frac{1}{\#F} \sum_{A \in F} |\det(\mathbb{1} - A_* D_*)|_\infty .
\end{aligned}$$

□

This averaging formula does not hold in general for almost-crystallographic groups. See propositions 7.1.6, 10.1.6 and 10.2.1 for counterexamples.

Finally, let us again consider dynamical zeta functions. Theorem 4.2.3 says that if the Reidemeister number is finite, it equals the Nielsen number. We immediately obtain the following implication for the zeta functions.

Corollary 4.2.7. *Let M be an infra-nilmanifold and $f : M \to M$ a self-map. If $R_f(z)$ exists, then $R_f(z) = N_f(z)$.*

The rationality of Reidemeister and Nielsen zeta functions on infra-nilmanifolds was first studied in [Won01], and was proven by Dekimpe and Dugardein in [DD15, Corollary 4.7].

Theorem 4.2.8. *Nielsen zeta functions on infra-nilmanifolds are rational.*

This theorem is proven by showing that a Nielsen zeta function $N_f(z)$ must always equal one of $L_f(z)$, $L_f(z)^{-1}$, $L_f(-z)$, $L_f(-z)^{-1}$, or the quotient of two Lefschetz zeta functions. Since Lefschetz zeta functions are always rational, the result follows.

Of course, since Reidemeister zeta functions coincide with Nielsen zeta functions, Reidemeister zeta functions on infra-nilmanifolds are always rational. This was also mentioned in [FL15, Proposition 3.2].

The proof of theorem 4.2.8 is topological in nature, since it uses the existence of Lefschetz numbers, Lefschetz zeta functions, etc. To the author's knowledge, Reidemeister-Nielsen theory has not been developed for orbifolds, hence these techniques cannot be used to research the rationality of Reidemeister zeta functions of almost-crystallographic groups.

Part II

Nilpotent groups

Chapter 5

Finitely generated, torsion-free, nilpotent groups

In this chapter, we will determine the (extended) Reidemeister spectra of certain finitely generated, torsion-free, nilpotent groups. Let us start by proving that any such group has ∞ in its Reidemeister spectrum, so that we may omit this calculation later.

Proposition 5.0.1. *Let* id_N *be the identity map on a finitely generated, torsion-free, nilpotent group* N*. Then* $R(\mathrm{id}_N) = \infty$*.*

Proof. If G is the Mal'cev completion of N, then $\mathrm{id}_G : G \to G$ is an affine homotopy lift of id_N, and the induced morphism on the associated Lie algebra $\mathfrak{g}$ is the identity as well. By applying theorem 4.1.5 we obtain that $R(\mathrm{id}_N) = \infty$. □

5.1 Abelian groups

A finitely generated, torsion-free, nilpotent group is abelian if and only if its nilpotency class is 1. Such group is isomorphic to the group $\mathbb{Z}^n$, where n is its Hirsch length. The Reidemeister spectra of these groups are well-known (see for example [Rom11, Section 3]), but for the sake of completeness we will prove them anyway.

Theorem 5.1.1. *The group* $\mathbb{Z}$ *has Reidemeister spectrum* $\{2, \infty\}$ *and full extended Reidemeister spectrum.*

Proof. Any endomorphism of $\mathbb{Z}$ is uniquely determined by the image of 1, since this element generates $\mathbb{Z}$. Let us use φ_n to denote the endomorphism satisfying $\varphi_n(1) = n$. From example 2.5.9 we know that

$$R(\varphi_n) = |1 - n|_\infty.$$

If $m \in \mathbb{N}$, then $R(\varphi_{1-m}) = m$ and $R(\varphi_1) = \infty$, hence $\mathbb{Z}$ has full extended Reidemeister spectrum. The only values of n for which φ_n is an automorphism are $n \in \{-1, 1\}$. These have Reidemeister numbers 2 and ∞ respectively, hence $\mathbb{Z}$ has Reidemeister spectrum $\{2, \infty\}$. $\square$

Theorem 5.1.2. *The groups $\mathbb{Z}^n$ with $n \geq 2$ have full (extended) Reidemeister spectrum.*

Proof. Consider the family of $n \times n$-matrices

$$D_m := \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & \ddots & & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & -m \end{pmatrix},$$

with $m \in \mathbb{N}$. By expanding the determinant of the matrix $\mathbb{1}_n - D_m$ along its n-th column, we obtain

$$\begin{aligned} \det(\mathbb{1}_n - D_m) &= \det \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \vdots & 0 \\ 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -1 & 1+m \end{pmatrix} \\ &= (-1)^{2n-1} + (-1)^{2n}(1+m) \\ &= m. \end{aligned}$$

Each matrix D_m is an automorphism of $\mathbb{Z}^n$, hence using the formula from example 2.5.9, we obtain that $\mathbb{Z}^n$ with $n \geq 2$ has full (extended) Reidemeister spectrum. $\square$

5.2 Nilpotent groups

To calculate the Reidemeister number of an automorphism of a finitely generated, torsion-free, nilpotent group, we will make use of theorem 4.1.6.

The extended Reidemeister spectrum can be calculated for any such group in a fairly straightforward way.

Theorem 5.2.1. *A finitely generated, torsion-free, nilpotent group has full extended Reidemeister spectrum.*

Proof. Let N be a finitely generated, torsion-free, nilpotent group. Since N is poly-$\mathbb{Z}$ (see theorem 3.1.13), there exists a normal subgroup $M \triangleleft N$ such that N is the semidirect product $N = M \rtimes_\psi \mathbb{Z}$. Let

$$p : M \rtimes_\psi \mathbb{Z} \to \mathbb{Z} : (m, z) \mapsto z,$$

$$i : \mathbb{Z} \to M \rtimes_\psi \mathbb{Z} : z \mapsto (1, z),$$

be the projection to and inclusion in the second factor respectively. From any endomorphism

$$\varphi_n : \mathbb{Z} \to \mathbb{Z} : z \mapsto nz,$$

we can construct an endomorphism $\phi_n := i \circ \varphi_n \circ p$ on $M \rtimes_\psi \mathbb{Z}$. Two elements $(m_1, z_1), (m_2, z_2)$ are ϕ_n-equivalent if and only if

$$\begin{aligned} (m_1, z_1) \sim_{\phi_n} (m_2, z_2) &\iff \exists (m', z') : (m_1, z_1) = (m', z')(m_2, z_2)\phi_n(m', z')^{-1} \\ &\iff \exists (m', z') : (m_1, z_1) = (m', z')(m_2, z_2)(1, nz')^{-1} \\ &\iff \exists (m', z') : (m_1, z_1) = (m'\psi_{z'}(m_2), z' + z_2 - nz'). \end{aligned}$$

Since we can always pick $m' = m_1 \psi_{z'}(m_2)^{-1}$, we continue with the following equivalences:

$$\begin{aligned} &\iff \exists z' : z_1 = z' + z_2 - nz' \\ &\iff z_1 \sim_{\varphi_n} z_2. \end{aligned}$$

This is just the φ_n-twisted conjugacy on $\mathbb{Z}$, so $R(\phi_n) = R(\varphi_n)$. From theorem 5.1.1 we then find that N has full extended Reidemeister spectrum. □

5.2.1 Dimension 3

A finitely generated, torsion-free, nilpotent group of dimension 3 can have nilpotency class at most 2. It is known (see [Seg83, Chapter 11, Proposition 5]) that such group is isomorphic to one of the groups

$$N_k = \langle a, b, c \mid [b, a] = c^k, [c, a] = [c, b] = 1 \rangle,$$

with $k \in \mathbb{N}$, which we already introduced in example 3.1.10. We also mentioned there that the adapted lower central series of N_k is given by

$$N_k \geq \langle c \rangle \geq 1.$$

Theorem 5.2.2 (see [Dug16, Theorem 9.2.8])**.** *Let N be a finitely generated, torsion-free, nilpotent group of rank 3 and nilpotency class 2. Then its Reidemeister spectrum is $2\mathbb{N} \cup \{\infty\}$.*

Proof. Assume that $N = N_k$ (as defined above) for some $k \in \mathbb{N}$. Let $\varphi \in \mathrm{Aut}(N)$ and assume that $R(\varphi) < \infty$. Then φ induces the automorphisms

$$(\varphi)_1 : N/\langle c \rangle \to N/\langle c \rangle, \qquad (\varphi)_2 : \langle c \rangle \to \langle c \rangle,$$

on the factors of its adapted lower central series. We know from theorem 4.1.6 that $R(\varphi) = R((\varphi)_1)R((\varphi)_2)$. Since $\langle c \rangle$ is isomorphic to $\mathbb{Z}$, theorem 5.1.1 tells us that $R((\varphi)_2) = 2$, and hence $R(\varphi) \in 2\mathbb{N}$.

Let us now prove that for every $m \in \mathbb{N}$, there exists an automorphism φ_m with $R(\varphi_m) = 2m$. Define φ_m as

$$\varphi_m(a) = b, \quad \varphi_m(b) = ab^{-m}, \quad \varphi_m(c) = c^{-1}.$$

This is a well-defined automorphism on N, since

$$\begin{aligned} [\varphi_m(b), \varphi_m(a)] &= b^m a^{-1} b^{-1} a b^{-m} b \\ &= b^m b^{-1} a^{-1} c^{-k} a b^{-m} b \\ &= b^m b^{-1} a^{-1} a b^{-m} b c^{-k} \\ &= c^{-k} \\ &= \varphi_m(c)^k. \end{aligned}$$

The automorphism $(\varphi_m)_1$ on $N/\langle c \rangle \cong \mathbb{Z}^2$ is the matrix

$$(\varphi_m)_1 = \begin{pmatrix} 0 & 1 \\ 1 & -m \end{pmatrix},$$

which has Reidemeister number $R((\varphi_m)_1) = m$ as proven in theorem 5.1.2. We also have that $R((\varphi_m)_2) = 2$, hence $R(\varphi_m) = 2m$ and thus $\mathrm{Spec}_R(N) = 2\mathbb{N} \cup \{\infty\}$. ∎

5.2.2 Dimension 4

A finitely generated, torsion-free, nilpotent group of dimension 4 can have nilpotency class at most 3. In [Dek96, Corollary 6.2.4], it is shown that such group with nilpotency class 2 is isomorphic to a group

$$N_k = \left\langle a, b, c, d \;\middle|\; \begin{array}{ll} [b,a] = d^k & [c,b] = 1 \\ [c,a] = 1 & [d,b] = 1 \\ [d,a] = 1 & [d,c] = 1 \end{array} \right\rangle,$$

for some $k \in \mathbb{N}$.

Theorem 5.2.3 (see [Dug16, Theorem 9.2.9]). *Let N be a finitely generated, torsion-free, nilpotent group of rank 4 and nilpotency class 2. Then its Reidemeister spectrum is $4\mathbb{N} \cup \{\infty\}$.*

Proof. Assume that $N = N_k$ (as defined above) for some $k \in \mathbb{N}$. Consider the central series given by

$$N \geq \langle c, d \mid [c,d] = 1\rangle \geq \langle d \rangle \geq 1.$$

The second group is exactly $Z(N)$ and the third is $\sqrt[N]{\gamma_2(N)}$, hence this central series satisfies the conditions needed to apply theorem 4.1.6. Let $\varphi \in \mathrm{Aut}(N)$ and assume that $R(\varphi) < \infty$. Then φ induces automorphisms

$$(\varphi)_1 : N/\langle c, d\rangle \to N/\langle c, d\rangle,$$

$$(\varphi)_2 : \langle c, d\rangle/\langle d\rangle \to \langle c, d\rangle/\langle d\rangle,$$

$$(\varphi)_3 : \langle d\rangle \to \langle d\rangle,$$

on the factors of the central series, and

$$R(\varphi) = R((\varphi)_1)R((\varphi)_2)R((\varphi)_3).$$

Since both $\langle c, d\rangle/\langle d\rangle$ and $\langle d\rangle$ are isomorphic to $\mathbb{Z}$, theorem 5.1.1 tells us that $R((\varphi)_2) = R((\varphi)_3) = 2$, and hence $R(\varphi) \in 4\mathbb{N}$.

Let us now prove that for every $m \in \mathbb{N}$, there exists an automorphism φ_m with $R(\varphi_m) = 4m$. Define φ_m as

$$\varphi_m(a) = b, \quad \varphi_m(b) = ab^{-m}, \quad \varphi_m(c) = c^{-1}, \quad \varphi_m(d) = d^{-1},$$

then just like in the proof of theorem 5.2.2, φ_m is a well-defined automorphism, and we can calculate that $R(\varphi_m) = 4m$. Thus $\mathrm{Spec}_R(N) = 4\mathbb{N} \cup \{\infty\}$. □

Next, let us consider the case of nilpotency class 3. In [Dek96, Proposition 6.2.6], it is shown that such group is isomorphic to

$$N_{k_1,k_2,k_3} = \left\langle a, b, c, d \mid \begin{array}{ll} [b,a] = c^{k_1} d^{k_2} & [c,b] = 1 \\ [c,a] = d^{k_3} & [c,d] = 1 \\ [d,a] = 1 & [d,c] = 1 \end{array} \right\rangle,$$

with $k_1, k_3 \in \mathbb{N}$ and $k_2 \in \mathbb{Z}$.

Theorem 5.2.4 (see [Dug16, Theorem 9.2.10]). *A finitely generated, torsion-free, nilpotent group of rank* 4 *and nilpotency class* 3 *has the* R_∞*-property.*

Proof. Assume that $N = N_{k_1,k_2,k_3}$ (as defined above) for some k_1, k_2, k_3. The adapted lower central series of such group N is given by

$$N \geq \langle c, d \mid [c,d] = 1 \rangle \geq \langle d \rangle \geq 1.$$

Let $\varphi \in \operatorname{Aut}(N)$ and assume that $R(\varphi) < \infty$. Then φ induces automorphisms

$$(\varphi)_1 : N/\langle c, d\rangle \to N/\langle c, d\rangle,$$

$$(\varphi)_2 : \langle c, d\rangle/\langle d\rangle \to \langle c, d\rangle/\langle d\rangle,$$

$$(\varphi)_3 : \langle d\rangle \to \langle d\rangle.$$

By theorem 4.1.6 we have that $R(\varphi) = R((\varphi)_1)R((\varphi)_2)R((\varphi)_3)$. Note that if $R((\varphi)_2) < \infty$ and $R((\varphi)_3) < \infty$, then $(\varphi)_2(c\langle d\rangle) = c^{-1}\langle d\rangle$ and $(\varphi)_3(d) = d^{-1}$. So in turn $\varphi(c) = c^{-1}d^\gamma$ for some $\gamma \in \mathbb{Z}$, and $\varphi(d) = d^{-1}$. Let us now set

$$\varphi(a) = a^{\alpha_1} b^{\alpha_2} c^{\alpha_3} d^{\alpha_4},$$

$$\varphi(b) = a^{\beta_1} b^{\beta_2} c^{\beta_3} d^{\beta_4}.$$

Since φ is an automorphism, one can compute that

$$d^{-k_3} = \varphi(d)^{k_3} = [\varphi(c), \varphi(a)] = d^{-k_3\alpha_1},$$

$$1 = [\varphi(c), \varphi(b)] = d^{-k_3\beta_1}.$$

Thus, $\alpha_1 = 1$ and $\beta_1 = 0$. But then the matrix corresponding to $(\varphi)_1$ is of the form

$$(\varphi)_1 = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}.$$

Using example 2.5.9 we then find that

$$R((\varphi)_1) = |\det(\mathbb{1}_2 - \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix})|_\infty = \infty,$$

hence also $R(\varphi) = \infty$. □

Chapter 6

Free nilpotent groups

Most of the results of this chapter can be found in [DTV17].

6.1 Reidemeister theory on free nilpotent groups

We would like to use theorem 4.1.6 to calculate the Reidemeister numbers of free nilpotent groups. Thus, it is essential for us to be able to compute the determinants $\det(\mathbb{1} - (\varphi)_i)$ for a given endomorphism φ of $N_{r,c}$. This is equivalent to understanding the eigenvalues of the matrices $(\varphi)_i$. The lemma below (which is a more explicit version of [DG14, Lemma 2.4]) shows that these are completely determined by the eigenvalues of $(\varphi)_1$.

Definition 6.1.1. Fix an r-tuple of complex numbers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$. Let $H = \cup_{n \in \mathbb{N}} H_n$ be a Hall basis of the free Lie algebra $\mathfrak{f}_r$. We define a map $f_\lambda : H \to \mathbb{C}$ inductively by

- $\forall i \in \{1, 2, \dots, r\}$: $f_\lambda(X_i) := \lambda_i$.

Let $n \geq 2$ and assume that $f_\lambda(X)$ has been defined for all $X \in H_k$ with $1 \leq k \leq n-1$.

- Consider $X \in H_n$, then $X = [U, V]$ for some $U \in H_k$ and $V \in H_l$ with $k + l = n$. We set $f_\lambda(X) := f_\lambda(U) f_\lambda(V)$.

We will say that f_λ is the map associated to λ.

Example 6.1.2. We have that

$$f_\lambda([X_i, X_j]) = \lambda_i\lambda_j \text{ and } f_\lambda([X_i, [X_j, X_k]] = \lambda_i\lambda_j\lambda_k.$$

Lemma 6.1.3. *Let $r \geq 2$ and $c \geq 1$ be positive integers and assume that $\varphi \in \mathrm{End}(N_{r,c})$ is an endomorphism inducing endomorphisms $(\varphi)_k$ on the quotients $\gamma_k(N_{r,c})/\gamma_{k+1}(N_{r,c})$ $(1 \leq k \leq c)$.*

Let $\lambda_1, \lambda_2, \cdots, \lambda_r$ be the eigenvalues of $(\varphi)_1$ (each eigenvalue is listed as many times as its multiplicity). Let H be a Hall basis of the free Lie algebra $\mathfrak{f}_r$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. Let $f_\lambda : H \to \mathbb{C}$ be the map associated to λ. Then the eigenvalues of $(\varphi)_k$ $(1 \leq k \leq c)$ are given by

$$\mathrm{Spec}((\varphi)_k) = \{f_\lambda(X) \mid X \in H_k\}.$$

In this way each eigenvalue is then listed as many times as its multiplicity.

Proof. Let φ_* denote the corresponding morphism on the Lie algebra $\mathfrak{g}_{r,c}$. As mentioned before, the eigenvalues of $(\varphi)_i$ are the same as the eigenvalues of $(\varphi_*)_i$, the morphism induced by φ_* on $\gamma_i(\mathfrak{g}_{r,c})/\gamma_{i+1}(\mathfrak{g}_{r,c})$ (as they can be represented by the same matrix). It is well known that the semisimple part of φ_* is also an automorphism of $\mathfrak{g}_{r,c}$ (See for example [Seg83, Corollary 2, page 135]) having the same eigenvalues as φ_* (also on each quotient $\gamma_i(\mathfrak{g}_{r,c})/\gamma_{i+1}(\mathfrak{g}_{r,c})$). Therefore, we may assume that φ_* is semisimple. Let $\mathfrak{g}_{r,c}^{\mathbb{C}} = \mathfrak{g}_{r,c} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}_{r,c}$, then there exists a basis of $\mathfrak{g}_{r,c}^{\mathbb{C}}$ consisting of eigenvectors for φ_* (which we can also consider as being a morphism of $\mathfrak{g}_{r,c}^{\mathbb{C}}$). It follows that we can find r eigenvectors $X_1, X_2, \ldots, X_r$ of $\mathfrak{g}_{r,c}^{\mathbb{C}}$ such that their canonical projections $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_r \in \mathfrak{g}_{r,c}^{\mathbb{C}}/\gamma_2(\mathfrak{g}_{r,c}^{\mathbb{C}})$ form a basis of $\mathfrak{g}_{r,c}^{\mathbb{C}}/\gamma_2(\mathfrak{g}_{r,c}^{\mathbb{C}})$. This implies that $X_1, X_2, \ldots, X_r$ are free generators of the free nilpotent Lie algebra $\mathfrak{g}_{r,c}^{\mathbb{C}}$. We can assume that H is a Hall basis with $H_1 = \{X_1, X_2, \ldots, X_r\}$ and that X_j is an eigenvector with eigenvalue λ_j. By induction, it now follows that if $X \in H_i$ $(1 \leq i \leq c)$, then X is an eigenvector for φ_* with eigenvalue $f_\lambda(X)$. Indeed, assume that $i \geq 2$ and the claim already holds for smaller values of i, then X is of the form $X = [U, V]$ with $U \in H_k$ and $V \in H_l$ for some $k, l < i$. Then

$$\begin{aligned}\varphi_*(X) &= \varphi_*([U, V]) = [\varphi_*(U), \varphi_*(V)] \\ &= [f_\lambda(U)U, f_\lambda(V)V] = f_\lambda(U)f_\lambda(V)[U, V] = f_\lambda(X)X.\end{aligned}$$

As the canonical projections of the vectors in H_i form a basis for the vector space $\gamma_i(\mathfrak{g}_{r,c}^{\mathbb{C}})/\gamma_{i+1}(\mathfrak{g}_{r,c}^{\mathbb{C}})$, it follows that the collection of eigenvalues of $(\varphi_*)_i$, and hence also of $(\varphi)_i$, is exactly the collection of values $f_\lambda(X)$, where X ranges over all vectors in H_i. $\square$

Example 6.1.4. Continuing example 3.2.27 and example 6.1.2 we find that when $\lambda_1, \lambda_2, \dots, \lambda_r$ are the eigenvalues of $(\varphi)_1$, then the eigenvalues of $(\varphi)_2$ are

$$\lambda_i \lambda_j \text{ with } 1 \leq i < j \leq r,$$

and those of $(\varphi)_3$ are

$$\lambda_i \lambda_j \lambda_k \text{ with } 1 \leq j < k \leq r \text{ and } 1 \leq j \leq i \leq r.$$

We are especially interested in the case that φ is an automorphism. In this case the induced map $(\varphi)_1$ will be an automorphism of $\mathbb{Z}^r$. We can consider the morphism

$$\psi : \mathrm{Aut}(N_{r,c}) \to \mathrm{Aut}(\mathbb{Z}^r) : \varphi \mapsto (\varphi)_1,$$

which is surjective. Indeed, it is well known that the analogous map $\mathrm{Aut}(F_r) \to \mathrm{Aut}(\mathbb{Z}^r)$ for the free group is surjective (see [MKS76, Theorem N4, Section 3.5]). Since all automorphisms of F_r induce an automorphism on $N_{r,c}$, the surjectivity of ψ follows immediately.

As explained above, $R(\varphi)$ only depends on the eigenvalues of φ_*, which are completely determined by the eigenvalues of $(\varphi)_1$ (by lemma 6.1.3). Hence, it is enough to know the characteristic polynomial of $(\varphi)_1$, which is of the form

$$p(x) = x^r + a_{r-1}x^{r-1} + \dots + a_1 x + a_0, \quad a_i \in \mathbb{Z}, \quad a_0 \in \{-1, 1\}, \tag{6.1}$$

since $a_0 = (-1)^r \det((\varphi)_1)$.

Conversely, any monic polynomial $p(x)$ of degree r of the form (6.1) (still with $a_i \in \mathbb{Z}$ and $a_0 = \pm 1$) is the characteristic polynomial of its companion matrix $C_p \in \mathrm{GL}_n(\mathbb{Z})$, where

$$C_p = \begin{pmatrix} & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_{r-1} \end{pmatrix}.$$

As ψ is surjective, we know that there exists an automorphism $\varphi \in \mathrm{Aut}(N_{r,c})$ with $(\varphi)_1 = C_p$. So instead of focusing on the automorphisms φ, we will in the sequel focus on the corresponding characteristic polynomial. Let $p(x)$ be a polynomial of the form (6.1). We will denote by $R_c(p(x))$ the Reidemeister number of any automorphism φ of $N_{r,c}$, such that the corresponding automorphism $(\varphi)_1$ has $p(x)$ as its characteristic polynomial.

Thus, in order to calculate the Reidemeister spectrum of $N_{r,c}$, we have to compute all possible numbers $R_c(p(x))$ for all possible polynomials $p(x)$.

6.1.1 Elementary symmetric polynomials

Our approach to calculating the Reidemeister numbers $R_c(p(x))$ makes use of the so-called elementary symmetric polynomials, i.e. the multivariate polynomials e_k defined as

$$e_k(x_1, x_2, \ldots, x_n) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

If $p(x)$ is any monic polynomial

$$p(x) = x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0,$$

with complex roots $\lambda_1, \lambda_2, \ldots, \lambda_r$, then $p(x)$ can be written as

$$p(x) = \prod_{i=1}^{r}(x - \lambda_i) = x^r + \sum_{i=1}^{r}(-1)^i e_i(\lambda_1, \lambda_2, \ldots, \lambda_r)x^{r-i},$$

or in terms of the coefficients this means that

$$a_{r-i} = (-1)^i e_i(\lambda_1, \lambda_2, \ldots, \lambda_r),$$

for all $1 \leq i \leq r$.

Theorem 6.1.5 (Fundamental theorem of elementary symmetric polynomials, see [Mac95, Section I.2])**.** *Let A be any commutative ring and $q(x_1, x_2, \ldots, x_n)$ a symmetric polynomial in $A[x_1, x_2, \ldots, x_n]$. Then there exists a polynomial $r(x_1, x_2, \ldots, x_n)$ in $A[x_1, x_2, \ldots, x_n]$ such that*

$$q(x_1, \ldots, x_n) = r(e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)).$$

This theorem tells us that if we have a symmetric polynomial $q \in \mathbb{Z}[\lambda_1, \ldots, \lambda_r]$ whose variables λ_i are the roots of a monic polynomial $p(x) = x^r + \sum_{i=0}^{r-1} a_i x^i$ with integral coefficients, then actually $q \in \mathbb{Z}[a_0, \ldots, a_{r-1}]$.

To calculate the Reidemeister spectrum of $N_{r,c}$, we will adopt a "divide and conquer" strategy, splitting up $R_c(p(x))$ in factors that are each symmetric polynomials in the roots λ_i, and calculating these factors in terms of the coefficients a_i.

Example 6.1.6. Let φ be an automorphism of $N_{r,c}$, let p be the characteristic polynomial of $(\varphi)_1$ and $\lambda_1, \ldots, \lambda_r$ the roots of p (and hence the eigenvalues of $(\varphi)_1$). Then

$$\det(\mathbb{1} - (\varphi)_1) = \prod_{i=1}^{r}(1 - \lambda_i) = p(1) = \sum_{i=0}^{r-1} a_i + 1.$$

6.1.2 Results on general free nilpotent groups

For a general free nilpotent group, there exists a sufficient (but not necessary) criterion such that its Reidemeister spectrum is not full.

Proposition 6.1.7. *Let $N_{r,c}$ be a free nilpotent group with $c \geq r$. Then the Reidemeister spectrum of $N_{r,c}$ is not full.*

Proof. For any polynomial $p(x)$, the r-th factor of $R_c(p(x))$ will be a product of the form

$$\prod(1 - \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r}).$$

Splitting this further up in factors that are each symmetric polynomials, one of them will be

$$(1 - \lambda_1\lambda_2\cdots\lambda_r) = 1 - a_0,$$

which is 0 or 2 depending on the constant term a_0. Hence either $R_c(p(x)) = \infty$ or $R_c(p(x)) \in 2\mathbb{N}$, and therefore $\mathrm{Spec}_R(N_{r,c}) \subseteq 2\mathbb{N} \cup \{\infty\}$. □

In section 6.2.2 we will show that the Reidemeister spectrum of $N_{3,2}$ is not full, illustrating that the criterion above is not necessary.

For the R_∞-property, there does exist a necessary and sufficient criterion. Let us first recall the following:

Lemma 6.1.8 (see [DG14, Proposition 2.3])**.** *For any $r \in \mathbb{N}$, there exists a matrix $A_r \in \mathrm{GL}_r(\mathbb{Z})$ with r distinct eigenvalues λ_1, λ_2, ..., λ_r such that*

$$\forall k \in \{1, 2, \ldots, 2r-1\}, \forall i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, r\} : \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k} \neq 1.$$

Theorem 6.1.9 (see [DG14, Theorem 2.5])**.** *A free nilpotent group $N_{r,c}$ has the R_∞ property if and only if $c \geq 2r$.*

Proof. We first prove that if $c \geq 2r$, then $R_c(p(x)) = \infty$. For any polynomial $p(x)$, the $2r$-th factor of $R_c(p(x))$ will be a product of the form

$$\prod(1 - \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{2r}}).$$

Splitting this further up in factors that are each symmetric polynomials, one of them will be

$$(1 - \lambda_1^2\lambda_2^2\cdots\lambda_r^2) = 1 - (a_0)^2 = 0,$$

hence $R_c(p(x)) = \infty$.

Conversely, let $c < 2r$ and let $p(x)$ be the characteristic polynomial of a matrix A_r as defined in lemma 6.1.8. Then $R_c(p(x))$ is the product of factors of the form

$$(1 - \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k})$$

with $k < 2r$. Because of the choice of A_r, all of these factors are non-zero and hence $R_c(p(x)) < \infty$. □

6.2 Nilpotency class 2

In this section we want to prove that the Reidemeister spectrum of $N_{r,2}$ is full for all $r \geq 4$, as well as calculate the spectra of $N_{2,2}$ and $N_{3,2}$. The Reidemeister number $R_2(p(x))$ is given by

$$R_2(p(x)) = \left| \prod_i (1 - \lambda_i) \prod_{i<j} (1 - \lambda_i\lambda_j) \right|_\infty .$$

6.2.1 Rank 2

We are dealing with a polynomial $p(x)$ given by

$$p(x) = x^2 + a_1 x + a_0,$$

with $a_1 \in \mathbb{Z}$ and $a_0 \in \{-1, 1\}$. The polynomial p has two roots λ_1, λ_2, for which $\lambda_1\lambda_2 = a_0$ and $\lambda_1 + \lambda_2 = -a_1$. As shown in example 6.1.6 the first product of $R_2(p(x))$ is $p(1) = 1 + a_1 + a_0$. The second product is $1 - \lambda_1\lambda_2 = 1 - a_0$. Thus,

$$R_2(p(x)) = |(a_0 + a_1 + 1)(1 - a_0)|_\infty .$$

If we assume that the Reidemeister number is finite, then we must have $a_0 = -1$. Then $R_2(p(x)) = 2|a_1|_\infty$ and hence $R_2(p(x)) \in 2\mathbb{N}$.

We define the family of polynomials $q_n(x) := x^2 + nx - 1$ with $n \in \mathbb{N}$, then $R_2(q_n(x)) = 2n$, hence $\mathrm{Spec}_R(N_{2,2}) = 2\mathbb{N} \cup \{\infty\}$. The result of this computation coincides with theorem 5.2.2 (since $N_{2,2}$ is isomorphic to N_1 as defined above said theorem), and with [Rom11, Section 3], where $\mathrm{Spec}_R(N_{2,2})$ was computed via other techniques.

6.2.2 Rank 3

We are dealing with a polynomial $p(x)$ given by

$$p(x) = x^3 + a_2x^2 + a_1x + a_0,$$

with $a_1, a_2 \in \mathbb{Z}$ and $a_0 \in \{-1, 1\}$. The polynomial p has three roots $\lambda_1, \lambda_2, \lambda_3$, for which

$$\begin{aligned} a_0 &= -\lambda_1\lambda_2\lambda_3, \\ a_1 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ a_2 &= -\lambda_1 - \lambda_2 - \lambda_3. \end{aligned}$$

The first product of $R_2(p(x))$ is $p(1) = 1 + a_2 + a_1 + a_0$. The second product can be calculated as follows:

$$\begin{aligned} \prod_{i<j}(1 - \lambda_i\lambda_j) &= \left[\prod_k \lambda_k\right]^{-1} \prod_{\substack{i\neq k\neq j \\ i<j}} (\lambda_k - \lambda_i\lambda_j\lambda_k) \\ &= -a_0^{-1} \prod_k (\lambda_k + a_0) \\ &= a_0^{-1} p(-a_0) \\ &= -a_0^2 + a_0a_2 - a_1 + 1 \\ &= a_0a_2 - a_1, \end{aligned}$$

where we used that $a_0^2 = 1$ in the last step. Hence we obtain

$$\begin{aligned} R_2(p(x)) &= |(1 + a_2 + a_1 + a_0)(a_0a_2 - a_1)|_\infty \\ &= \begin{cases} |a_2 + a_1|_\infty^2 & \text{if } a_0 = -1, \\ |(a_2 + 1)^2 - (a_1 + 1)^2|_\infty & \text{if } a_0 = 1. \end{cases} \end{aligned}$$

Thus $R_2(p(x))$ is a square or the difference of two squares, so it must be a multiple of four or an odd number. We define two families of polynomials. First, set $q_n(x) := x^3 + nx^2 + (n-1)x + 1$ with $n \in \mathbb{N}$, then $R_2(q_n(x)) = 2n + 1$. Second, set $r_n(x) := x^3 + nx^2 + (n-2)x + 1$ with $n \in \mathbb{N}$, then $R_2(r_n(x)) = 4n$. Hence $\mathrm{Spec}_R(N_{3,2}) = (2\mathbb{N} - 1) \cup 4\mathbb{N} \cup \{\infty\}$. Again, this result coincides with that of [Rom11, Section 3].

6.2.3 Even rank at least 4

Let $r = 2m$ for some $m \in \mathbb{N}$ with $m \geq 2$, and let $n \in \mathbb{N}$ be arbitrary. For the polynomial

$$p_{2m,n}(x) = x^{2m} - x^{m+1} + (n-1)x^m + 1$$

with roots $\lambda_1, \lambda_2, \ldots, \lambda_{2m}$, we will show that $R_2(p_{2m,n}(x)) = n$. This polynomial was first considered in [Mij14] where it was also conjectured that indeed $R_2(p_{2m,n}(x)) = n$. In her thesis M. Mijle verified this conjecture for $m = 2, 3, \ldots, 9$.

In the computations below, we will simply write $p(x)$ instead of $p_{2m,n}(x)$. The first factor in the computation of $R_2(p(x))$ is $p(1) = n$ (see example 6.1.6), so it suffices to prove that

$$\left[\prod_{i<j}(1 - \lambda_i\lambda_j)\right]^2 = \prod_{i \neq j}(1 - \lambda_i\lambda_j) = 1. \tag{6.2}$$

We note that $\prod_i \lambda_i = 1$ because $p(x)$ has even degree and has constant term equal to 1. Also, if λ_i is a root of $p(x)$ then

$$\lambda_i^{2m} + (n-1)\lambda_i^m + 1 = \lambda_i^{m+1},$$

so

$$\begin{aligned}
\lambda_i^{2m} p(\lambda_i^{-1}) &= \lambda_i^{2m} + (n-1)\lambda_i^m - \lambda_i^{m-1} + 1 \\
&= \lambda_i^{m+1} - \lambda_i^{m-1} \\
&= -\lambda_i^{m-1}(1 - \lambda_i^2),
\end{aligned}$$

giving

$$p(\lambda_i^{-1}) = -\lambda_i^{-m-1}(1 - \lambda_i^2). \tag{6.3}$$

We want a polynomial whose roots include $\lambda_i\lambda_j$, so to this end we define

$$q(x) := \prod_i p\left(\frac{x}{\lambda_i}\right).$$

We then calculate

$$\begin{aligned} q(x) &= \prod_i p\left(\frac{x}{\lambda_i}\right) \\ &= \prod_{i,j}\left(\frac{x}{\lambda_i} - \lambda_j\right) \\ &= \left[\prod_i \lambda_i\right]^{-2m} \prod_{i,j}(x - \lambda_i\lambda_j) \\ &= \prod_{i,j}(x - \lambda_i\lambda_j) \\ &= \prod_{i\neq j}(x - \lambda_i\lambda_j)\prod_i(x - \lambda_i^2), \end{aligned} \tag{6.4}$$

where in the second-to-last line we used the noted fact that $\prod_i \lambda_i = 1$. Consider

$$q(1) = \prod_{i\neq j}(1 - \lambda_i\lambda_j)\prod_i(1 - \lambda_i^2). \tag{6.5}$$

For comparison, from (6.3) we obtain an alternate representation for $q(1)$:

$$\begin{aligned} q(1) &= \prod_i p\left(\lambda_i^{-1}\right) \\ &= \prod_i \left[-\lambda_i^{-m-1}(1 - \lambda_i^2)\right] \\ &= \left[\prod_i \lambda_i\right]^{-m-1} \prod_i(1 - \lambda_i^2) \\ &= \prod_i(1 - \lambda_i^2), \end{aligned} \tag{6.6}$$

where we have again used the fact that $\prod_i \lambda_i = 1$. Since $n \in \mathbb{N}$ and $p(1) = n$, 1 is not a root of p.

If -1 is not a root of p, then $\lambda_i^2 \neq 1$ for all i, so the factor $\prod_i(1 - \lambda_i^2)$ in both (6.5) and (6.6) is non-zero. The desired identity (6.2) then follows.

Now suppose that -1 is a root of p, then $0 = p(-1) = 2 + n(-1)^m$ and hence $n = 2(-1)^{m+1}$. For the derivative p' of p we find that

$$\begin{aligned} p'(-1) &= 2m(-1)^{2m-1} - (m+1)(-1)^m + m(n-1)(-1)^{m-1} \\ &= -2m + (m+1)(-1)^{m+1} + m(2(-1)^{m+1} - 1)(-1)^{m-1} \\ &= (-1)^{m+1}. \end{aligned} \tag{6.7}$$

Hence -1 is not a double root, so we can call this root λ_1. For $x \neq 1$ we can divide both sides of (6.4) by $x - 1$ to get

$$\frac{q(x)}{x-1} = \prod_{i \neq j}(x - \lambda_i\lambda_j)\prod_{i>1}(x - \lambda_i^2),$$

and hence

$$\lim_{x\to 1}\frac{q(x)}{x-1} = \prod_{i\neq j}(1 - \lambda_i\lambda_j)\prod_{i>1}(1-\lambda_i^2). \tag{6.8}$$

Alternatively,

$$\frac{q(x)}{x-1} = \frac{p(-x)}{x-1}\prod_{i>1} p\left(\frac{x}{\lambda_i}\right),$$

so that using l'Hôpital's rule we find

$$\begin{aligned} \lim_{x\to 1}\frac{q(x)}{x-1} &= \frac{d}{dx}p(-x)\bigg|_{x=1}\prod_{i>1}p(\lambda_i^{-1}) \\ &= -p'(-1)\prod_{i>1}\left[-\lambda_i^{-m-1}(1-\lambda_i^2)\right] \\ &= p'(-1)\left[\prod_{i>1}\lambda_i\right]^{-m-1}\prod_{i>1}(1-\lambda_i^2) \\ &= \prod_{i>1}(1-\lambda_i^2), \end{aligned} \tag{6.9}$$

where in the second line we used identity (6.3) and in the last line we used both (6.7) and the fact that

$$\prod_{i>1}\lambda_i = \lambda_1^{-1} = \lambda_1 = -1.$$

By comparing (6.8) and (6.9) the desired identity (6.2) follows.

As a conclusion of this computation we find:

Theorem 6.2.1. *Let $m \geq 2$ be an integer, then $\mathrm{Spec}_R(N_{2m,2})$ is full.*

6.2.4 Odd rank at least 5

Let $r = 2m + 1$ for some $m \in \mathbb{N}$ with $m \geq 2$, and let $n \in \mathbb{N}$ be arbitrary. For the polynomial

$$\begin{aligned} p_{2m+1,n}(x) := x^{2m+1} + (n+1)x^{m+2} + (1-n)x^{m+1} \\ + (n-1)x^m - nx^{m-1} - 1, \end{aligned}$$

with roots $\lambda_1, \lambda_2, \dots, \lambda_{2m+1}$, we will show that $R_2(p_{2m+1,n}(x)) = n + c(m)$, where

$$\begin{aligned} c(m) &= 2 + \cos\left(m\frac{\pi}{3}\right) + \sqrt{3}\sin\left(m\frac{\pi}{3}\right) \\ &= \begin{cases} 0 & \text{if } m \equiv 4 \pmod 6, \\ 1 & \text{if } m \equiv 3 \pmod 6 \text{ or } m \equiv 5 \pmod 6, \\ 3 & \text{if } m \equiv 0 \pmod 6 \text{ or } m \equiv 2 \pmod 6, \\ 4 & \text{if } m \equiv 1 \pmod 6. \end{cases} \end{aligned}$$

It then follows that $R_2(p_{2m+1,n-c(m)}(x)) = n$. The proof uses similar techniques as for the case where r is even. Again, during the computations, we will simply write $p(x)$ instead of $p_{2m+1,n}(x)$.

As always, the first factor of $R_2(p(x))$ is $p(1) = 1$, so it suffices to prove that

$$\left|\prod_{i<j}(1 - \lambda_i\lambda_j)\right| = n + c(m). \tag{6.10}$$

We first calculate some specific values of $p(x)$:

$$p(1) = \prod_i (1 - \lambda_i) = 1, \tag{6.11}$$

$$p(-1) = -\prod_i (1 + \lambda_i) = (-1)^m(4n - 1) - 2. \tag{6.12}$$

We find that both 1 and -1 are not roots of $p(x)$. Also, for any root λ_i we have

$$-\lambda_i^{2m+1} = (n+1)\lambda_i^{m+2} + (1-n)\lambda_i^{m+1} + (n-1)\lambda_i^m - n\lambda_i^{m-1} - 1,$$

so

$$\begin{aligned}
\lambda_i^{2m+1}p(\lambda_i^{-1}) &= -\lambda_i^{2m+1} - n\lambda_i^{m+2} + (n-1)\lambda_i^{m+1} \\
&\quad + (1-n)\lambda_i^{m} + (n+1)\lambda_i^{m-1} + 1 \\
&= \lambda_i^{m+2} + \lambda_i^{m-1} \\
&= \lambda_i^{m-1}\left(1+\lambda_i^3\right) \\
&= \lambda_i^{m-1}\left(1+\lambda_i\right)\left(e^{\frac{\pi}{3}i} - \lambda_i\right)\left(e^{-\frac{\pi}{3}i} - \lambda_i\right),
\end{aligned}$$

giving

$$p(\lambda_i^{-1}) = \lambda_i^{-m-2}\left(1+\lambda_i\right)\left(e^{\frac{\pi}{3}i} - \lambda_i\right)\left(e^{-\frac{\pi}{3}i} - \lambda_i\right). \tag{6.13}$$

Again, we define a new polynomial $q(x)$ as

$$q(x) := \prod_i p\left(\frac{x}{\lambda_i}\right),$$

and once again

$$q(x) = \prod_{i\neq j}(x - \lambda_i\lambda_j)\prod_i(x-\lambda_i^2).$$

Let us evaluate $q(x)$ in $x = 1$:

$$\begin{aligned}
q(1) &= \prod_{i\neq j}\left(1-\lambda_i\lambda_j\right)\prod_i\left(1-\lambda_i^2\right) \\
&= \prod_{i\neq j}\left(1-\lambda_i\lambda_j\right)\prod_i\left(1-\lambda_i\right)\prod_i\left(1+\lambda_i\right) \\
&= -p(-1)\prod_{i\neq j}\left(1-\lambda_i\lambda_j\right),
\end{aligned} \tag{6.14}$$

where we used (6.11) and (6.12) in the last step.

We evaluate $q(x)$ in $x=1$ again, this time using (6.13):

$$\begin{aligned}
q(1) &= \prod_i p(\lambda_i^{-1}) \\
&= \prod_i \lambda_i^{-m-2}\,(1+\lambda_i)\left(e^{\frac{\pi}{3}i}-\lambda_i\right)\left(e^{-\frac{\pi}{3}i}-\lambda_i\right) \\
&= \left[\prod_i \lambda_j\right]^{-m-2} \prod_i (1+\lambda_i) \prod_i \left(e^{\frac{\pi}{3}i}-\lambda_i\right) \prod_i \left(e^{-\frac{\pi}{3}i}-\lambda_i\right) \\
&= -p(-1)p\left(e^{\frac{\pi}{3}i}\right)p\left(e^{-\frac{\pi}{3}i}\right) \\
&= -p(-1)\left|p\left(e^{\frac{\pi}{3}i}\right)\right|^2 .
\end{aligned} \tag{6.15}$$

Comparing equations (6.14) and (6.15) now gives

$$\prod_{i\neq j}(1-\lambda_i\lambda_j) = \left|p\left(e^{\frac{\pi}{3}i}\right)\right|^2,$$

or by using that the product on the left-hand side is symmetric in the indices, that

$$\left|\prod_{i<j}(1-\lambda_i\lambda_j)\right| = \left|p\left(e^{\frac{\pi}{3}i}\right)\right|.$$

One may now evaluate $\left|p\left(e^{\frac{\pi}{3}i}\right)\right|$ to obtain

$$\left|p\left(e^{\frac{\pi}{3}i}\right)\right| = \left|n+2+\cos\left(m\frac{\pi}{3}\right)+\sqrt{3}\sin\left(m\frac{\pi}{3}\right)\right| = n+c(m).$$

This proves the following theorem:

Theorem 6.2.2. *Let $m \geq 2$ be an integer, then $\mathrm{Spec}_R(N_{2m+1,2})$ is full.*

6.3 Rank 2

We are dealing with a polynomial $p(x)$ given by

$$p(x) = x^2 + a_1 x + a_0,$$

with $a_1 \in \mathbb{Z}$ and $a_0 \in \{-1, 1\}$. The polynomial p has two roots λ_1, λ_2, for which $\lambda_1\lambda_2 = a_0$ and $\lambda_1 + \lambda_2 = -a_1$. The spectrum of $N_{2,2}$ has already been calculated in the section on nilpotency class 2.

6.3.1 Nilpotency class 3

If the nilpotency class c is 3, then $R_3(p(x))$ is given by

$$R_3(p(x)) = \left| \prod_{i}(1-\lambda_i) \prod_{i<j}(1-\lambda_i\lambda_j) \prod_{\substack{j<k\\ j\leq i}}(1-\lambda_i\lambda_j\lambda_k) \right|_\infty .$$

Again, let us assume this Reidemeister number is finite. From the calculations we made for $N_{2,2}$, we know that $\lambda_1\lambda_2 = a_0 = -1$ and that the first two products are a_1 and 2 respectively. The third product becomes

$$\prod_{\substack{j<k\\ j\leq i}}(1-\lambda_i\lambda_j\lambda_k) = (1-\lambda_1^2\lambda_2)(1-\lambda_1\lambda_2^2) = (1+\lambda_1)(1+\lambda_2) = p(-1) = a_1,$$

so all factors combined give $R_3(p(x)) = 2|a_1|_\infty^2$, hence $R_3(p(x)) \in 2\mathbb{N}^2$.

Defining the polynomials $q_n := x^2+nx-1$ with $n \in \mathbb{N}$, we find $R_3(q_n(x)) = 2n^2$, hence $\mathrm{Spec}_R(N_{2,3}) = 2\mathbb{N}^2 \cup \{\infty\}$. This result was also obtained, using another approach, in [Rom11, Section 3].

6.4 Rank 3

We are dealing with a polynomial $p(x)$ given by

$$p(x) = x^3 + a_2x^2 + a_1x + a_0,$$

with $a_1, a_2 \in \mathbb{Z}$ and $a_0 \in \{-1, 1\}$. The polynomial p has three roots $\lambda_1, \lambda_2, \lambda_3$, for which

$$a_0 = -\lambda_1\lambda_2\lambda_3,$$

$$a_1 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,$$

$$a_2 = -\lambda_1 - \lambda_2 - \lambda_3.$$

We have already computed the spectrum of $N_{3,2}$ in the section on nilpotency class 2.

6.4.1 Nilpotency class 3

If the nilpotency class c is 3, then $R_3(p(x))$ is given by

$$R_3(p(x)) = \left| \prod_i (1-\lambda_i) \prod_{i<j} (1-\lambda_i\lambda_j) \prod_{\substack{j<k \\ j\leq i}} (1-\lambda_i\lambda_j\lambda_k) \right|_\infty .$$

From our calculations for $N_{3,2}$, we already know the first two products. We can split the third product in two symmetric polynomials in the λ_i:

$$\prod_{\substack{j<k \\ j\leq i}} (1-\lambda_i\lambda_j\lambda_k) = (1-\lambda_1\lambda_2\lambda_3)^2 \prod_{i\neq j} (1-\lambda_i^2\lambda_j).$$

The first polynomial equals $(1+a_0)^2$. If we assume that $R_3(p(x))$ is finite, then $a_0 = -\lambda_1\lambda_2\lambda_3$ must equal 1, and then this polynomial equals 4. Let us tackle the second polynomial.

$$\begin{aligned}\prod_{i\neq j} (1-\lambda_i^2\lambda_j) &= \left[\prod_k \lambda_k\right]^{-2} \prod_{i\neq j\neq k\neq i} \left(\lambda_k - \lambda_i^2\lambda_j\lambda_k\right) \\ &= (-1)^{-2} \prod_{i\neq k} (\lambda_k + \lambda_i) \\ &= \left[\prod_j (-a_2 - \lambda_j)\right]^2 \\ &= p(-a_2)^2 \\ &= (1-a_1a_2)^2.\end{aligned}$$

Putting everything together we find

$$R_3(p(x)) = 4\left|(2+a_1+a_2)(a_1-a_2)(1-a_1a_2)^2\right|_\infty .$$

Substituting $a = 1 + a_1$ and $b = 1 + a_2$, we may rewrite this as

$$R_3(p(x)) = 4|(a^2-b^2)(a+b-ab)^2|_\infty,$$

and in particular, if $a \neq \pm b$, then $|a^2-b^2| \geq |a|+|b|$, hence $R_3(p(x)) \geq 4(|a|+|b|)$. This allows us to, in some sense, calculate the Reidemeister spectrum of $N_{3,3}$.

By calculating $R_3(p(x))$ for all pairs (a,b) with $|a| < |b| \leq 250$, we know that the Reidemeister numbers less than 1000 are exactly 4, 12, 20, 32, 60, 64, 96, 108, 140, 192, 252, 300, 320, 324, 396, 480, 500, 572, 672, 700, 756, 780, 800, 896 and 980.

To give a general idea on what numbers can be in the spectrum, consider the different values of a and b mod 2:

- $a, b \equiv 0 \bmod 2$. Then $|a^2 - b^2|$ is a multiple of 4 and $a + b - ab$ is a multiple of 2. Hence $R_3(p(x)) \in 64\mathbb{N}$.
- $a \equiv 0, b \equiv 1 \bmod 2$ or vice versa. Then both $|a^2 - b^2|$ and $a + b - ab$ are odd. Hence $R_3(p(x)) \in 4(2\mathbb{N} - 1)$.
- $a, b \equiv 1 \bmod 2$. Then $|a^2 - b^2|$ is a multiple of 8 and $a + b - ab$ is odd. Hence $R_3(p(x)) \in 32\mathbb{N}$.

Together with the calculated Reidemeister numbers mentioned earlier, we may then state that $\mathrm{Spec}_R(N_{3,3}) \subsetneq 32\mathbb{N} \cup 4(2\mathbb{N} - 1) \cup \{\infty\}$.

6.4.2 Nilpotency class 4

If the nilpotency class c is 4, we require a total order on H_2 (the elements of length 2 in the Hall basis). Let us use $<$ to denote the lexicographic order on $\mathbb{N}^2$, i.e. $(i,j) < (k,l)$ if and only if $i < k$ or $i = k$ and $j < l$. We then put a total order on H_2 by saying that $[X_i, X_j] < [X_k, X_l]$ if and only if $(i,j) < (k,l)$. The elements of H_4 are then given by

- $[X_i, [X_j, [X_k, X_l]]]$ with $k < l$, $k \leq j \leq i$,
- $[[X_i, X_j], [X_k, X_l]]$ with $i < j$, $k < l$, $(i,j) < (k,l)$.

Thus $R_4(p(x))$ is given by

$$R_4(p(x)) = R_3(p(x)) \left| \prod_{\substack{k \leq j \leq i \\ k < l}} (1 - \lambda_i \lambda_j \lambda_k \lambda_l) \prod_{\substack{i<j \\ k<l \\ (i,j)<(k,l)}} (1 - \lambda_i \lambda_j \lambda_k \lambda_l) \right|_\infty .$$

Since the rank is 3, the two products at the end can be rewritten as the following three products, based on the number of distinct roots in every factor.

$$\prod_{i \neq j} (1 - \lambda_i^3 \lambda_j) \prod_{i<j} (1 - \lambda_i^2 \lambda_j^2) \prod_{\substack{j \neq i \neq k \\ j<k}} (1 - \lambda_i^2 \lambda_j \lambda_k)^3.$$

Because we are interested in finite Reidemeister numbers, we must assume that $a_0 = -\lambda_1\lambda_2\lambda_3 = 1$. We start with the first product.

$$\begin{aligned}
\prod_{i\neq j}(1-\lambda_i^3\lambda_j) &= \left[\prod_k \lambda_k\right]^{-2} \prod_{i\neq j\neq k\neq i}(\lambda_k - \lambda_i^3\lambda_j\lambda_k) \\
&= (-1)^{-2}\prod_{i\neq k}(\lambda_k+\lambda_i^2) \\
&= \left[\prod_i(\lambda_i+\lambda_i^2)\right]^{-1}\prod_{i,k}(\lambda_k+\lambda_i^2) \\
&= \left[\prod_i \lambda_i\right]^{-1}\left[\prod_i(1+\lambda_i)\right]^{-1}\prod_{i,k}(i\sqrt{\lambda_k}-\lambda_i)(-i\sqrt{\lambda_k}-\lambda_i) \\
&= p(-1)^{-1}\prod_k p(i\sqrt{\lambda_k})p(-i\sqrt{\lambda_k}).
\end{aligned}$$

We now calculate what $p(i\sqrt{\lambda_k})p(-i\sqrt{\lambda_k})$ is for any root λ_k of f:

$$\begin{aligned}
p(i\sqrt{\lambda_k})p(-i\sqrt{\lambda_k}) &= \lambda_k^3 + (a_2^2-2a_1)\lambda_k^2 + (a_1^2-2a_2)\lambda_k + 1 \\
&= (a_2^2 - a_2 - 2a_1)\lambda_k^2 + (a_1^2 - a_1 - 2a_2)\lambda_k \\
&= (a_2^2 - a_2 - 2a_1)\lambda_k\left(\lambda_k + \frac{a_1^2-a_1-2a_2}{a_2^2-a_2-2a_1}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\prod_{i\neq j}(1-\lambda_i^3\lambda_j) &= p(-1)^{-1}(a_2^2-a_2-2a_1)^3\prod_k \lambda_k\left(\lambda_k + \frac{a_1^2-a_1-2a_2}{a_2^2-a_2-2a_1}\right) \\
&= \frac{(a_2^2-a_2-2a_1)^3}{a_2-a_1}p\left(-\frac{a_1^2-a_1-2a_2}{a_2^2-a_2-2a_1}\right) \\
&= -a_1^5 - a_2^5 - 3a_1^4a_2 - 3a_1a_2^4 - a_1^4 - a_2^4 + a_1^3a_2^3 \\
&\quad + 5a_1^3a_2 + 5a_1a_2^3 + a_1^3 + a_2^3 + 10a_1^2a_2^2 - a_1^2a_2 \\
&\quad - a_1a_2^2 - 7a_1^2 - 7a_2^2 - 13a_1a_2.
\end{aligned}$$

The second product is less troublesome:

$$\begin{aligned}\prod_{i<j}(1-\lambda_i^2\lambda_j^2) &= \left[\prod_{k=1}^{3}\lambda_k\right]^{-2}\prod_{\substack{i\neq k\neq j\\ i<j}}(\lambda_k^2-\lambda_i^2\lambda_j^2\lambda_k^2)\\
&= (-1)^{-2}\prod_k(\lambda_k^2-1)\\
&= \prod_k(1-\lambda_k)(-1-\lambda_k)\\
&= p(1)p(-1)\\
&= (a_1+a_2+2)(a_2-a_1).\end{aligned}$$

Finally, for the third product, we have:

$$\begin{aligned}\prod_{\substack{j\neq i\neq k\\ j<k}}(1-\lambda_i^2\lambda_j\lambda_k)^3 &= \prod_i(1+\lambda_i)^3\\
&= -p(-1)^3\\
&= (a_1-a_2)^3.\end{aligned}$$

All products together give us that

$$R_4(p(x)) = 4\left|(a_1-a_2)^5(a_1+a_2+2)^2(1-a_1a_2)^2(a_1^5+\cdots+13a_1a_2)\right|_\infty,$$

or if we again substitute $a=a_1+1$ and $b=a_2+1$, we get

$$R_4(p(x)) = 4\left|(a-b)^3(a^2-b^2)^2(a+b-ab)^2(a^5+\cdots-94ab)\right|_\infty.$$

One can show that every such Reidemeister number is divisible by 32. The 10 smallest Reidemeister numbers are

(1) 32,

(2) 288,

(3) 416,

(4) 6400,

(5) 8192,

(6) 69984,

(7) 139264,

(8) 559872,

(9) 980000,

(10) 1138688.

6.4.3 Nilpotency class 5

If the nilpotency class c is 5, then the elements of H_5 are given by

- $[X_i, [X_j, [X_k, [X_l, X_m]]]]$ with $l < m$, $l \leq k \leq j \leq i$,
- $[[X_i, X_j], [X_k, [X_l, X_m]]]$ with $i < j$, $l < m$, $l \leq k$.

Thus $R_5(p(x))$ is given by

$$R_5(p(x)) = R_4(p(x)) \left| \prod_{\substack{l \leq k \leq j \leq i \\ l < m}} (1 - \lambda_i \lambda_j \lambda_k \lambda_l \lambda_m) \prod_{\substack{i < j \\ l < m \\ l \leq k}} (1 - \lambda_i \lambda_j \lambda_k \lambda_l \lambda_m) \right|_\infty .$$

Since the rank is 3, the two products at the end can be rewritten as the following four products, based on the number of distinct roots in every factor.

$$\prod_{i \neq j} (1 - \lambda_i^4 \lambda_j) \prod_{i \neq j} (1 - \lambda_i^3 \lambda_j^2)^2 \prod_{\substack{j \neq i \neq k \\ j < k}} (1 - \lambda_i^3 \lambda_j \lambda_k)^4 \prod_{\substack{i \neq k \neq j \\ i < j}} (1 - \lambda_i^2 \lambda_j^2 \lambda_k)^6.$$

Because we are interested in finite Reidemeister numbers, we must assume that $a_0 = -\lambda_1 \lambda_2 \lambda_3 = -1$. The first factor is once again the hardest to express in terms of a_1 and a_2. We will work as follows:

$$\begin{aligned}
\prod_{i\neq j}(1-\lambda_i^4\lambda_j) &= \left[\prod_{k=1}^{3}\lambda_k\right]^{-2}\prod_{i\neq j\neq k\neq i}(\lambda_k-\lambda_i^4\lambda_j\lambda_k)\\
&= (-1)^{-2}\prod_{i\neq k}(\lambda_k+\lambda_i^3)\\
&= \left[\prod_i(\lambda_i+\lambda_i^3)\right]^{-1}\prod_{i,k}(\lambda_k+\lambda_i^3)\\
&= -\left[\prod_i\lambda_i(i-\lambda_i)(-i-\lambda_i)\right]^{-1}\\
&\quad\cdot\prod_{i,k}(-\sqrt[3]{\lambda_k}-\lambda_i)(e^{\frac{\pi}{3}i}\sqrt[3]{\lambda_k}-\lambda_i)(e^{-\frac{\pi}{3}i}\sqrt[3]{\lambda_k}-\lambda_i)\\
&= -\left[\prod_i\lambda_i\right]^{-1}p(i)^{-1}p(-i)^{-1}\\
&\quad\cdot\prod_k p(-\sqrt[3]{\lambda_k})p\left(e^{\frac{\pi}{3}i}\sqrt[3]{\lambda_k}\right)p\left(e^{-\frac{\pi}{3}i}\sqrt[3]{\lambda_k}\right)\\
&= |p(i)|^{-2}\prod_k p(-\sqrt[3]{\lambda_k})p\left(e^{\frac{\pi}{3}i}\sqrt[3]{\lambda_k}\right)p\left(e^{-\frac{\pi}{3}i}\sqrt[3]{\lambda_k}\right).
\end{aligned}$$

For any root λ_k of p, the expression $p(-\sqrt[3]{\lambda_k})p\left(e^{\frac{\pi}{3}i}\sqrt[3]{\lambda_k}\right)p\left(e^{-\frac{\pi}{3}i}\sqrt[3]{\lambda_k}\right)$ equals

$$\begin{aligned}
&-\lambda_k^3+(a_2^3-3a_1a_2+3)\lambda_k^2+(-a_1^3+3a_1a_2-3)\lambda_k+1\\
&\quad=(a_2^3-3a_1a_2+a_2+3)\lambda_k^2+(-a_1^3+3a_1a_2+a_1-3)\lambda_k+2\\
&\quad=A\left(\lambda_k+\frac{B+\sqrt{D}}{2A}\right)\left(\lambda_k+\frac{B-\sqrt{D}}{2A}\right),
\end{aligned}$$

with

$$A = a_2^3 - 3a_1a_2 + a_2 + 3,$$

$$B = -a_1^3 + 3a_1a_2 + a_1 - 3,$$

$$D = -8a_2^3 + 9a_1^2a_2^2 - 6a_1^4a_2 + 6a_1^2a_2 + 6a_1a_2 - 8a_2 + a_1^6 - 2a_1^4 + 6a_1^3 + a_1^2 - 6a_1 - 15.$$

Hence:

$$\begin{aligned}\prod_{i\neq j}(1-\lambda_i^4\lambda_j) &= |p(i)|^{-2}A^3\prod_k\left(\lambda_k+\frac{B+\sqrt{D}}{2A}\right)\left(\lambda_k+\frac{B-\sqrt{D}}{2A}\right)\\ &= \frac{A^3}{(a_1^2+a_2^2-2a_1-2a_2+2)}p\left(-\frac{B+\sqrt{D}}{2A}\right)p\left(-\frac{B-\sqrt{D}}{2A}\right)\\ &= a_1^7+a_2^7+2a_1^6+2a_2^6-4a_1^5a_2^2-4a_1^2a_2^5-7a_1^5a_2-7a_1a_2^5+a_1^5\\ &\quad + a_2^5 + a_1^4a_2^4 - 7a_1^4a_2 - 7a_1a_2^4 + 7a_1^4 + 7a_2^4 + 17a_1^3a_2^3\\ &\quad + 14a_1^3a_2^2 + 14a_1^2a_2^3 - 10a_1^3a_2 - 10a_1a_2^3 + 11a_1^3 + 11a_2^3\\ &\quad - 19a_1^2a_2^2 - 22a_1^2a_2 - 22a_1a_2^2 + 11a_1^2 + 11a_2^2 - a_1a_2 + 10a_1\\ &\quad + 10a_2 + 4.\end{aligned}$$

The second product is still somewhat troublesome:

$$\begin{aligned}\prod_{i\neq j}(1-\lambda_i^3\lambda_j^2) &= \left[\prod_k \lambda_k\right]^{-2} \prod_{i\neq j\neq k\neq i}(\lambda_k^2-\lambda_i^3\lambda_j^2\lambda_k^2)\\ &= (-1)^{-2}\prod_{i\neq k}(\lambda_k^2-\lambda_i)\\ &= \left[\prod_i(\lambda_k^2-\lambda_k)\right]^{-1}\prod_{i,k}(\lambda_k^2-\lambda_i)\\ &= \left[\prod_i \lambda_k(\lambda_k-1)\right]^{-1}\prod_{i,k}(\sqrt{\lambda_i}-\lambda_k)(-\sqrt{\lambda_i}-\lambda_k)\\ &= -p(1)^{-1}\prod_i p(\sqrt{\lambda_i})p(-\sqrt{\lambda_i}).\end{aligned}$$

We now calculate what $p(\sqrt{\lambda_i})p(-\sqrt{\lambda_i})$ is for any root λ_k of p:

$$\begin{aligned}f(\sqrt{\lambda_i})f(-\sqrt{\lambda_i}) &= -\lambda_i^3+(-2a_1+a_2^2)\lambda_i^2+(-a_1^2+2a_2)\lambda_i+1\\ &= (-2a_1+a_2^2+a_2)\lambda_i^2+(-a_1^2+a_1+2a_2)\lambda_i+2\\ &= A\left(\lambda_i+\frac{B+\sqrt{D}}{2A}\right)\left(\lambda_i+\frac{B-\sqrt{D}}{2A}\right),\end{aligned}$$

with

$$A = -2a_1+a_2^2+a_2,$$

$$B = -a_1^2+a_1+2a_2,$$

$$D = a_1^4-2a_1^3-4a_1^2a_2+a_1^2+4a_1a_2+16a_1-4a_2^2-8a_2.$$

Hence:

$$\prod_{i\neq j}(1-\lambda_i^3\lambda_j^2)^2 = -p(1)^{-1}A^3\prod_i\left(\lambda_i+\frac{B+\sqrt{D}}{2A}\right)\left(\lambda_i+\frac{B-\sqrt{D}}{2A}\right)$$
$$= -\frac{A^3}{a_1+a_2+2}p\left(-\frac{B+\sqrt{D}}{2A}\right)p\left(-\frac{B-\sqrt{D}}{2A}\right)$$
$$= a_1^5+a_2^5-3a_1^4a_2-3a_1a_2^4+a_1^4+a_2^4+a_1^3a_2^3-5a_1^3a_2$$
$$-5a_1a_2^3+a_1^3+a_2^3+10a_1^2a_2^2+3a_1^2a_2+3a_1a_2^2$$
$$+3a_1^2+3a_2^2-13a_1a_2-2a_1-2a_2+4.$$

The third product is not too hard.

$$\prod_{\substack{j\neq i\neq k\\ j<k}}(1-\lambda_i^3\lambda_j\lambda_k) = \prod_i(1+\lambda_i^2)$$
$$= \prod_i(i-\lambda_i)(-i-\lambda_i)$$
$$= p(i)p(-i)$$
$$= |p(i)|^2$$
$$= a_1^2+a_2^2-2a_1-2a_2+2.$$

Finally, the fourth product is quite easy as well.

$$\prod_{\substack{i\neq k\neq j\\ i<j}}(1-\lambda_i^2\lambda_j^2\lambda_k) = \left[\prod_{k=1}^3\lambda_k\right]^{-1}\prod_{\substack{i\neq k\neq j\\ i<j}}(\lambda_k-\lambda_i^2\lambda_j^2\lambda_k^2)$$
$$= -\prod_k(\lambda_k-1)$$
$$= -p(1)$$
$$= -a_1-a_2-2.$$

All products together give us that

$$R_5(p(x)) = 4\left|(a_1-a_2)^5(a_1+a_2+2)^8(1-a_1a_2)^2(a_1^2+\cdots+2)^4\right.$$
$$\left.(a_1^5+\cdots+13a_1a_2)(a_1^5+\cdots+4)^2(a_1^7+\cdots+4)\right|_\infty,$$

or if we again substitute $a = a_1 + 1$ and $b = a_2 + 1$, we get

$$R_5(p(x)) = 4\left|(a+b)^3(a^2-b^2)^5(a+b-ab)^2(a^2+\cdots+8)^4\right.$$
$$\left.(a^5+\cdots-94ab)(a^5+\cdots+18ab)^2(a^7+\cdots+18ab)\right|_\infty.$$

One can show that every such Reidemeister number is divisible by 2048. The 10 smallest Reidemeister numbers are

(1) 1280000,
(2) 631535616,
(3) 9885304832,
(4) 646400000000,
(5) 11433202941952,
(6) 2304141516914688,
(7) 23464505849675776,
(8) 84943913980852224,
(9) 173876382269440000,
(10) 973098408800000000.

6.5 Direct products of free nilpotent groups

This section builds greatly on results obtained by K. Godecharle, as part of her Master's thesis [God16]. The goal of this thesis was to determine which direct products $N_{r,c} \times N_{r',c'}$ of two free nilpotent groups have the R_∞-property.

The aim of this section is to generalise this to a general direct product N of free nilpotent groups, and to obtain results for the Reidemeister spectrum as well. This product N can be written as

$$N = \prod_{i=1}^{m}\prod_{j=1}^{n_i} N_{r_i,c_i},$$

where $n_i \in \mathbb{N}$ and $(r_i, c_i) \neq (r_j, c_j)$ if $i \neq j$. Moreover, by combining the abelian factors ($\mathbb{Z}^r \times \mathbb{Z}^s = \mathbb{Z}^{r+s}$), we may assume that at most one factor is abelian.

First, let us set some notation. We define the maps

$$\iota_{i,j} : N_{r_i,c_i} \to N : x \mapsto (1, \dots, 1, \underbrace{x}_{j\text{-th factor } N_{r_i,c_i}}, 1, \dots, 1),$$

$$p_{i,j} : N \to N : (x_{1,1}, \dots, x_{i,j}, \dots, x_{m,n_m}) \mapsto (1, \dots, 1, x_{i,j}, 1, \dots, 1).$$

Note that any automorphism $\varphi \in \operatorname{Aut}(N)$ is completely determined by the morphisms

$$(\varphi \circ \iota_{i,j}) : N_{r_i,c_i} \to N.$$

The following proposition is a direct consequence of combining [God16, Lemma 4.1.1], [God16, Lemma 4.2.1] and [God16, Lemma 4.3.2].

Proposition 6.5.1. *For any $i \in \{1, \dots, m\}$, there exists a permutation $\sigma_i \in \mathcal{S}_{n_i}$ such that for every $j \in \{1, \dots, n_i\}$, we have that*

$$\operatorname{im}(\varphi \circ \iota_{i,j}) \subseteq Z_{1,1} \times \dots \times Z_{i,\sigma_i(j)-1} \times N_{r_i,c_i} \times Z_{i,\sigma_i(j)+1} \times \dots \times Z_{n,n_n},$$

where

$$Z_{k,l} = \begin{cases} Z(N_{r_k,c_k}) & \text{if } c_k \leq c_i \text{ or } c_i = 1, \\ \gamma_{c_i}(N_{r_k,c_k}) & \text{if } c_k > c_i > 1. \end{cases}$$

The next proposition again follows from results by Godecharle, in particular the proofs of [God16, Theorem 4.1.2], [God16, Theorem 4.3.3], [God16, Theorem 4.3.1] and [God16, Theorem 4.3.4].

Proposition 6.5.2. *Let φ be an automorphism of a direct product N of free nilpotent groups, and let σ_i be the permutations as defined in proposition 6.5.1. Define $\bar{\varphi}$ as the morphism such that*

$$p_{i,\sigma_i(j)} \circ \varphi \circ \iota_{i,j} = \bar{\varphi} \circ \iota_{i,j}.$$

Then $\bar{\varphi}$ is an automorphism of N and

$$(\varphi)_k = (\bar{\varphi})_k$$

for all $k = 1, \dots, c$, with $c = \max\{c_1, \dots, c_m\}$ the nilpotency class of N.

From the above and theorem 4.1.6, me way conclude that $R(\varphi) = R(\bar{\varphi})$. At the same time, we may now decompose $\bar{\varphi}$ as a product. Let $\sigma_{i,1}, \dots, \sigma_{i,l_i}$ be the cycles appearing in the disjoint cycle notation of σ_i, then

$$\bar{\varphi} = \prod_{i=1}^{m} \prod_{j=1}^{l_i} \varphi_{i,j},$$

with

$$\varphi_{i,j} = \sigma_{i,j} \circ (\varphi_{i,j,1} \times \varphi_{i,j,2} \times \cdots \times \varphi_{i,j,\#\sigma_{i,j}}),$$

where the $\varphi_{i,j,k}$ are automorphisms of N_{r_i,c_i}. Thus,

$$R(\bar{\varphi}) = \prod_{i=1}^{m} \prod_{j=1}^{l_i} R(\varphi_{i,j}).$$

It then suffices to calculate Reidemeister numbers $R(\varphi_{i,j})$. Because a Reidemeister number is invariant under conjugation by an automorphism, we may assume that

$$\begin{aligned} \sigma_{i,j} &= (\#\sigma_{i,j} \quad 1 \quad 2 \quad \cdots \quad \#\sigma_{i,j} - 1), \\ \varphi_{i,j} &= \sigma_{i,j} \circ (\psi_{i,j} \times \mathrm{id} \times \cdots \times \mathrm{id}), \end{aligned}$$

with $\psi_{i,j} \in \mathrm{Aut}(N_{r_i,c_i})$. We will now show that

$$R(\varphi_{i,j}) = R(\psi_{i,j}).$$

For any $k = 1, \dots, c$, we have that $(\varphi_{i,j})_k$ is of the form

$$\begin{pmatrix} & \mathbb{1} & & \\ & & \ddots & \\ & & & \mathbb{1} \\ A & & & \end{pmatrix},$$

for some invertible matrix A with integral coefficients. Now,

$$\begin{aligned} R((\varphi_{i,j})_k) &= |\det(\mathbb{1} - \begin{pmatrix} & \mathbb{1} & & \\ & & \ddots & \\ & & & \mathbb{1} \\ A & & & \end{pmatrix})|_\infty \\ &= |\det \begin{pmatrix} \mathbb{1} & -\mathbb{1} & & \\ & \ddots & \ddots & \\ & & \ddots & -\mathbb{1} \\ -A & & & \mathbb{1} \end{pmatrix} |_\infty. \end{aligned}$$

Row operations do not change the determinant of a matrix. So add the bottom row of block matrix to the second to last, then add the (new) second to last row to the third to last, and repeat until we add the new second row to the first row. We then get

$$\begin{aligned} R((\varphi_{i,j})_k) &= |\det \begin{pmatrix} \mathbb{1} - A & & & \\ -A & \mathbb{1} & & \\ \vdots & & \ddots & \\ -A & & & \mathbb{1} \end{pmatrix} |_\infty \\ &= |\det(\mathbb{1} - A)|_\infty \\ &= R((\psi_{i,j})_k), \end{aligned}$$

and hence $R(\varphi_{i,j}) = R(\psi_{i,j})$. We thus obtain that

$$R(\varphi) = \prod_{i=1}^{m} \prod_{j=1}^{l_i} R(\psi_{i,j}).$$

The following theorem now follows.

Theorem 6.5.3. *Let N be the direct product of free nilpotent groups N_{r_i,c_i}. Then N has the R_∞-property if and only if at least one of the factors N_{r_i,c_i} has the R_∞-property.*

Remark 6.5.4. In [God16, Section 4.2.1], it is claimed that the direct products $N_{2,4} \times N_{2,4}$ and $N_{2,5} \times N_{2,5}$ do not have the R_∞-property; an explicit automorphism φ with (supposedly) finite Reidemeister number is given. However, there is a mistake in the calculation of the eigenvalues of $(\varphi)_4$, which falsely implies that $R((\varphi)_4) < \infty$ and hence $R(\varphi) < \infty$.

Theorem 6.5.5. *Consider a direct product N of identical free nilpotent groups $N_{r,c}$, say*

$$N = \prod_{i=1}^{m} N_{r,c},$$

with $c > 1$. Then the spectrum of N is given by

$$\operatorname{Spec}_R(N) = \bigcup_{i=1}^{m} \left\{ \prod_{j=1}^{i} R_j \;\middle|\; R_j \in \operatorname{Spec}_R(N_{r,c}) \right\}.$$

Theorem 6.5.6. *Consider a direct product N of free nilpotent groups given by*

$$N = \prod_{i=1}^{m} \prod_{j=1}^{n_i} N_{r_i,c_i},$$

with $(r_i, c_i) \neq (r_j, c_j)$ if $i \neq j$, and at most one factor N_{r_i,c_i} is abelian. Then the spectrum of N is given by

$$\mathrm{Spec}_R(N) = \left\{ \prod_{i=1}^{m} R_i \;\middle|\; R_i \in \mathrm{Spec}_R\left(\prod_{j=1}^{n_i} N_{r_i,c_i} \right) \right\}.$$

Example 6.5.7. Let us consider some easy examples of Reidemeister spectra of direct products of free nilpotent groups.

(1) Let $N = \prod_{i=1}^{m} N_{2,2}$ for some $m \in \mathbb{N}$. Then the Reidemeister spectrum of N is given by
$$\mathrm{Spec}_R(N) = 2\mathbb{N} \cup \{\infty\}.$$

(2) Let $N = \prod_{i=1}^{m} N_{2,3}$ for some $m \geq 2$. Then the Reidemeister spectrum of N is given by
$$\mathrm{Spec}_R(N) = 2\mathbb{N}^2 \cup 4\mathbb{N}^2 \cup \{\infty\}.$$

(3) Let $N = N_{2,2} \times N_{2,3}$. Then the Reidemeister spectrum of N is given by
$$\mathrm{Spec}_R(N) = 4\mathbb{N}^2 \cup \{\infty\}.$$

(4) Let $N = N_{3,2} \times \mathbb{Z}^2$. Then the Reidemeister spectrum of N is full.

Part III

Almost-crystallographic groups

Chapter 7

Crystallographic groups with diagonal holonomy $\mathbb{Z}_2$

In this chapter, we consider crystallographic groups with diagonal holonomy $F \cong \mathbb{Z}_2$, i.e. the crystallographic groups which are isomorphic to a group

$$\Lambda_{n/k/\epsilon} := \left\langle \mathbb{Z}^n, \left(\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \epsilon/2 \end{pmatrix}, \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_{n-k} \end{pmatrix} \right) \right\rangle,$$

with $n \in \mathbb{N}$, $1 \leq k \leq n$ and $\epsilon \in \{0, 1\}$. Note that $\Lambda_{n/n/1} \cong \Lambda_{n/n/0}$, all other choices of parameters give rise to non-isomorphic groups. We have already encountered two such groups in example 3.3.2, namely $\Lambda_{1/1/0}$ and $\Lambda_{2/1/1}$, the (orbifold) fundamental groups of the closed interval and the Klein bottle respectively.

The following proposition gives us more insight in the structure of these groups.

Proposition 7.0.1. *The group $\Lambda := \Lambda_{n/k/\epsilon}$ has characteristic subgroups*

- $\sqrt[\Lambda]{\gamma_2(\Lambda)} = \langle e_1, \ldots, e_k \rangle$,
- $Z(\Lambda) = \langle e_{k+1}, \ldots, e_n \rangle$.

In particular, if $\epsilon = 0$, then

$$\Lambda \cong \Lambda_{k/k/0} \times \mathbb{Z}^{n-k},$$

with both factors characteristic.

7.1 The R_∞-property and Reidemeister spectrum

Most of the results in this section were published in [DKT19].

Just like for the finitely generated, torsion-free, nilpotent groups, let us start by proving that any almost-crystallographic group has ∞ in its Reidemeister spectrum, so that we may omit this calculation later.

Proposition 7.1.1. *Let* id_Γ *be the identity morphism on an almost-crystallographic group* Γ*. Then* $R(\mathrm{id}_\Gamma) = \infty$*.*

Proof. The restriction of id_Γ to the translation subgroup N is of course the identity id_N. In proposition 5.0.1 we have proven that $R(\mathrm{id}_N) = \infty$, hence by lemma 2.5.10(2) we find that $R(\mathrm{id}_\Gamma) = \infty$ as well. □

7.1.1 Non-Bieberbach groups

First, we will study the groups $\Lambda_{n/k/\epsilon}$ that are not torsion-free, i.e. those with $\epsilon = 0$. Using proposition 7.0.1, lemma 2.5.18 and the results obtained in section 5.1, it suffices to calculate the Reidemeister spectra of the groups $\Lambda_{k/k/0} \cong \langle \mathbb{Z}^k, (0, -\mathbb{1}_k)\rangle$. Let us first determine the automorphism group of $\Lambda_{k/k/0}$.

Proposition 7.1.2. *Let* $\Lambda := \Lambda_{k/k/0}$*. Then the map*

$$\Phi : \mathbb{Z}^k \rtimes \mathrm{GL}_k(\mathbb{Z}) \to \mathrm{Aut}(\Lambda) : (d, D) \mapsto \xi_{(d/2,D)}$$

is an isomorphism.

Proof. First, let us confirm the map is well-defined, i.e.

$$(d/2, D)(x, \pm\mathbb{1}_k)(d/2, D)^{-1} \in \Lambda$$

for any $x \in \mathbb{Z}^k$, where $D \in \mathrm{GL}_k(\mathbb{Z})$ and $d \in \mathbb{Z}^k$. We have that

$$(d/2, D)(x, \mathbb{1}_k)(d/2, D)^{-1} = (Dx, \mathbb{1}_k),$$

$$(d/2, D)(x, -\mathbb{1}_k)(d/2, D)^{-1} = (Dx + d, -\mathbb{1}_k),$$

hence the map is indeed well-defined. Second, it is straightforward to see that it is a group homomorphism. Finally, to prove that it is actually an isomorphism, we will give a homomorphism Ψ that is both a left and right inverse of Φ, i.e.

$$\Phi \circ \Psi = \mathrm{id}_{\mathrm{Aut}(\Lambda)}, \qquad \Psi \circ \Phi = \mathrm{id}_{\mathbb{Z}^k \rtimes \mathrm{GL}_k(\mathbb{Z})}\,.$$

If $\varphi = \xi_{(d,D)} \in \operatorname{Aut}(\Lambda)$, then

$$(d,D)(x,\mathbb{1}_k)(d,D)^{-1} = (Dx,\mathbb{1}_k),$$

$$(d,D)(x,-\mathbb{1}_k)(d,D)^{-1} = (Dx+2d,-\mathbb{1}_k),$$

hence we must have that $D \in \mathrm{GL}_k(\mathbb{Z})$ and $2d \in \mathbb{Z}^k$. Thus, the map

$$\Psi : \operatorname{Aut}(\Lambda) \to \mathbb{Z}^k \rtimes \mathrm{GL}_k(\mathbb{Z}) : \xi_{(d,D)} \mapsto (2d, D)$$

is both a left and right inverse to Φ, which is therefore an isomorphism. □

The following theorem gives us the Reidemeister spectrum of $\Lambda_{k/k/0}$.

Theorem 7.1.3. *Let $\Lambda_{k/k/0} \cong \langle \mathbb{Z}^k, (0,-\mathbb{1}_k)\rangle$. Then*

$$\operatorname{Spec}_R(\Lambda_{k/k/0}) = \begin{cases} \{\infty\} & \text{if } k=1,\\ 2\mathbb{N} \cup \{3,\infty\} & \text{if } k=2,\\ \mathbb{N}\setminus\{1\}\cup\{\infty\} & \text{if } k \geq 3. \end{cases}$$

The proof of this theorem is far from straightforward. We will first introduce some lemmas and intermediate results.

Lemma 7.1.4. *Let $B \in \mathbb{Z}^{n\times n}$ and $b \in \mathbb{Z}^n$. Define $O(B,b)$ as the number of solutions $\bar{x}$ over $\mathbb{Z}_2$ of the linear system of equations $\bar{B}\bar{x} = \bar{b}$, where the bar-notation stands for the element-wise projection to $\mathbb{Z}_2$. Then we have the following:*

- *when* $\det(B)$ *is odd,* $O(B,b) = 1$ *(so is also odd),*
- *when* $\det(B)$ *is even,* $O(B,b) = 0,2,4,\dots,2^n$ *(so is also even).*

Lemma 7.1.5. *Define an equivalence relation on $\mathbb{Z}^n$, determined by a matrix $B \in \mathbb{Z}^{n\times n}$ and an element $b \in \mathbb{Z}^n$, where*

$$\forall x,y \in \mathbb{Z}^n : x \sim y \iff \exists z \in \mathbb{Z}^n : x-y = Bz \text{ or } x+y+b = Bz.$$

The number of equivalence classes is then given by

$$E(B,b) = \frac{|\det(B)|_\infty + O(B,b)}{2}.$$

Proof. It is obvious from the definition that

$$x \sim y \iff y \in x + \operatorname{im}(B) \text{ or } y \in -x-b+\operatorname{im}(B).$$

From this it follows easily that the equivalence class of x, denoted by $[x]_\sim$, equals

$$[x]_\sim = (x + \mathrm{im}(B)) \cup (-x - b + \mathrm{im}(B)). \tag{7.1}$$

Moreover we have that either $(x+\mathrm{im}(B))\cap(-x-b+\mathrm{im}(B)) = \varnothing$ or $x+\mathrm{im}(B) = -x - b + \mathrm{im}(B)$. From example 2.5.9 we know there are $|\det(B)|_\infty$ cosets of $\mathrm{im}(B)$.

In general, elements x and $-x - b$ will not belong to the same coset of $\mathrm{im}(B)$ and the union in (7.1) will be a disjoint union. Let N denote the number of cosets $x + \mathrm{im}(B)$ such that $x + \mathrm{im}(B) = -x - b + \mathrm{im}(B)$. Then these N cosets form N equivalence classes for the relation $\sim$, while the other $|\det(B)|_\infty - N$ cosets come in pairs $(x+\mathrm{im}(B), -x-b+\mathrm{im}(B))$ and so determine the remaining $(|\det(B)|_\infty - N)/2$ equivalence classes of $\sim$. Therefore

$$E(B, b) = \frac{|\det(B)|_\infty - N}{2} + N = \frac{|\det(B)|_\infty + N}{2}.$$

We now determine this number N. We have that $x + \mathrm{im}(B)$ and $-x - b + \mathrm{im}(B)$ are actually the same coset if and only if

$$\exists z \in \mathbb{Z}^n : 2x + b = Bz. \tag{7.2}$$

We have to count for how many cosets $x + \mathrm{im}(B)$ this equation holds. For it to hold, it must definitely do so over $\mathbb{Z}_2$, i.e. $\bar{B}\bar{z} = \bar{b}$. So we have $O(B, b)$ solutions $\bar{z}$ over $\mathbb{Z}_2$. Next, we show that each solution $\bar{z}$ over $\mathbb{Z}_2$ produces a unique coset $x + \mathrm{im}(B)$ satisfying equation (7.2). Let $\bar{z}$ be a solution of $\bar{B}\bar{z} = \bar{b}$. Choose any lift $z \in \mathbb{Z}^n$ of $\bar{z}$, then $Bz - b \in 2\mathbb{Z}^n$, so there exists a unique $x \in \mathbb{Z}^n$ such that $Bz - b = 2x$. Hence for this x we have that equation (7.2) holds and so $x + \mathrm{im}(B) = -x - b + \mathrm{im}(B)$. However, the x we found depends on the choice of the lift z. Let $z' \in \mathbb{Z}^n$ be another element projecting down to $\bar{z}$ (so there exists a $c \in \mathbb{Z}^n$ with $z - z' = 2c$) and giving rise to x' satisfying $2x' = Bz' + b$. Then

$$\begin{aligned} 2(x - x') = Bz - b - (Bz' - b) = 2Bc &\implies x - x' = Bc \\ &\implies x + \mathrm{im}(B) = x' + \mathrm{im}(B), \end{aligned}$$

from which we see that the choice of the lift z is of no influence on the coset $x + \mathrm{im}(B)$: while x and x' may be different, they are both representatives of one and the same coset.

Hence every solution $\bar{z}$ gives rise to a unique coset with representative x satisfying equation (7.2). Note that if $\det(B) \neq 0$, each solution $\bar{z}$ produces a different coset: suppose by contradiction that two different solutions $\bar{z}_1$ and $\bar{z}_2$

produce the same coset $x+\operatorname{im}(B)$. This means there exist $z_1, z_2 \in \mathbb{Z}^n$, with $z_1 \neq z_2$, such that $2x+b = Bz_1 = Bz_2$, but then $B(z_1 - z_2) = 0$ and therefore $\det(B) = 0$, which we assumed was not the case. So the number N of cosets $x + \operatorname{im}(B)$ satisfying equation (7.2) is exactly $O(B,b)$ when $\det(B) \neq 0$. So in case $\det(B) \neq 0$, we have that

$$E(B,b) = \frac{|\det(B)|_\infty + O(B,b)}{2}.$$

If $\det(B) = 0$, there are infinitely many cosets $x + \operatorname{im}(B)$, so there are infinitely many pairs of disjoint cosets that together form one equivalence class, and at most $O(B,b)$ cosets that form an equivalence class on their own. Hence $E(B,b)$ is infinite and the formula above also holds in this case. □

Proposition 7.1.6. *Let $\Lambda = \langle \mathbb{Z}^n, (0, -\mathbb{1}_n)\rangle$ and $\varphi = \xi_{(d,D)} \in \operatorname{Aut}(\Lambda)$. Then the Reidemeister number of φ is given by*

$$R(\varphi) = \left(\frac{1}{\#F} \sum_{A \in F} |\det(\mathbb{1}_n - AD)|_\infty \right) + O(\mathbb{1}_n - D, 2d). \tag{7.3}$$

Proof. The holonomy group of Λ is given by $F = \{\pm \mathbb{1}_n\} = Z(\mathrm{GL}_n(\mathbb{Z}))$. Let $\varphi = \xi_{(d,D)}$ be an automorphism. Recall from proposition 7.1.2 that necessarily $d \in \left(\frac{1}{2}\mathbb{Z}\right)^n$, whereas D can be any matrix in $\mathrm{GL}_n(\mathbb{Z})$. Two elements $(x, A_x), (y, A_y) \in \Lambda$ are Reidemeister equivalent if and only if there exists an element $(z, A_z) \in \Lambda$ such that

$$\begin{aligned}
(y, A_y) &= (z, A_z)(x, A_x)\varphi(z, A_z)^{-1} \\
&= (z, A_z)(x, A_x)(d, D)(z, A_z)^{-1}(d, D)^{-1} \\
&= (z + A_z x + A_z A_x d - A_x D z - A_x d, A_z A_x D A_z^{-1} D^{-1}) \\
&= (A_z x + (\mathbb{1}_n - A_x D) z - (\mathbb{1}_n - A_z)(A_x d), A_x).
\end{aligned}$$

Thus a necessary requirement for (x, A_x) to be equivalent to (y, A_y) is that $A_x = A_y$. So an element $(x, \mathbb{1}_n)$ and an element $(y, -\mathbb{1}_n)$ can never be in the same Reidemeister class, and in particular $R(\varphi) \geq 2$. Now for two elements $(x, A), (y, A)$ with the same holonomy part A, $(x, A) \sim (y, A)$ if and only if there exists some $z \in \mathbb{Z}^n$ such that

$$x - y = (\mathbb{1}_n - AD)z \text{ or } x + y + 2Ad = (\mathbb{1}_n - AD)z,$$

where the first case corresponds to $A_z = \mathbb{1}_n$ and the second case to $A_z = -\mathbb{1}_n$. From the definition of $E(B, b)$ in lemma 7.1.5, we obtain that

$$\begin{aligned} R(\varphi) &= E(\mathbb{1}_n - D, 2d) + E(\mathbb{1}_n + D, -2d) \\ &= \frac{|\det(\mathbb{1}_n - D)|_\infty + O(\mathbb{1}_n - D, 2d)}{2} \\ &\quad + \frac{|\det(\mathbb{1}_n + D)|_\infty + O(\mathbb{1}_n + D, -2d)}{2}. \end{aligned}$$

But over $\mathbb{Z}_2$, $\overline{\mathbb{1}_n - D} = \overline{\mathbb{1}_n + D}$ and $\overline{2d} = \overline{-2d}$, hence

$$O(\mathbb{1}_n - D, 2d) = O(\mathbb{1}_n + D, -2d).$$

So we find the proposed formula:

$$R(\varphi) = \frac{|\det(\mathbb{1}_n - D)|_\infty + |\det(\mathbb{1}_n + D)|_\infty}{2} + O(\mathbb{1}_n - D, 2d). \qquad \square$$

Proof of theorem 7.1.3. We will use the formula from proposition 7.1.6. Also, recall from lemma 7.1.4 that $O(B, b)$ is odd (in fact, it then necessarily equals 1) if and only if $\det(B)$ is odd.

First, let us consider $n = 1$. Then either $D = 1$ or $D = -1$, and $\det(1 - D)$ or $\det(1 + D)$ vanishes respectively, hence $R(\varphi) = \infty$ and thus $\mathrm{Spec}_R(\Lambda) = \{\infty\}$.

Next, we deal with the case $n = 2$. Since $\det(\mathbb{1}_2 \pm D) = 1 \pm \mathrm{tr}(D) + \det(D)$ and $\det(D) = \pm 1$, we have that

$$\det(\mathbb{1}_2 \pm D) \equiv \mathrm{tr}(D) \equiv O(\mathbb{1}_2 - D, 2d) \pmod 2.$$

We now determine the value of $R(\varphi)$:

1. $\det(D) = -1$. Then the formula becomes

   $$R(\varphi) = |\,\mathrm{tr}(D)|_\infty + O(\mathbb{1}_2 - D, 2d).$$

   Depending on the value of $|\,\mathrm{tr}(D)|$, we have:

   (a) $|\,\mathrm{tr}(D)| = 0$, then $R(\varphi) = \infty$,

   (b) $|\,\mathrm{tr}(D)| \geq 1$, then $R(\varphi) = |\,\mathrm{tr}(D)| + O(\mathbb{1}_2 - D, 2d) \in 2\mathbb{N}$.

2. $\det(D) = 1$. Then the formula becomes

   $$R(\varphi) = \frac{|2 - \mathrm{tr}(D)|_\infty + |2 + \mathrm{tr}(D)|_\infty}{2} + O(\mathbb{1}_2 - D, 2d).$$

   Depending on the value of $|\,\mathrm{tr}(D)|$, we have:

(a) $|\operatorname{tr}(D)| = 0$, then $R(\varphi) = 2 + O(\mathbb{1}_2 - D, 2d) \in 2\mathbb{N}$,
(b) $|\operatorname{tr}(D)| = 1$, then $R(\varphi) = 3$,
(c) $|\operatorname{tr}(D)| = 2$, then $R(\varphi) = \infty$,
(d) $|\operatorname{tr}(D)| \geq 3$, then $R(\varphi) = |\operatorname{tr}(D)| + O(\mathbb{1}_2 - D, 2d) \in 2\mathbb{N}$.

So indeed $\operatorname{Spec}_R(\Lambda) \subseteq 2\mathbb{N} \cup \{3, \infty\}$. We now show that all these Reidemeister numbers can actually be attained. To obtain an even Reidemeister number, consider $\varphi_m = \xi_{(d,D_m)}$ with

$$D_m = \begin{pmatrix} 0 & 1 \\ 1 & 2m \end{pmatrix}, d = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},$$

with $m \in \mathbb{N}$, then $|\det(\mathbb{1}_2 - D_m)|_\infty = |\det(\mathbb{1}_2 + D_m)|_\infty = 2|m|$ and $O(\mathbb{1}_2 - D_m, 2d) = 0$, and hence $R(\varphi_m) = 2|m|$. Finally, to obtain Reidemeister number 3, consider $\varphi = \xi_{(0,D)}$ with

$$D = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

then $R(\varphi) = 3$. Hence $\operatorname{Spec}_R(\Lambda) = 2\mathbb{N} \cup \{3, \infty\}$.

Finally, consider the case $n \geq 3$. As mentioned in the proof of proposition 7.1.6, $R(\varphi) \geq 2$. We show that every natural number greater than or equal to 2 can be attained. Consider $\varphi_m = \xi_{(0,D_m)}$ with

$$D_m = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & \ddots & & & \vdots & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & 0 \\ \vdots & & \ddots & \ddots & 0 & m \\ 0 & \cdots & \cdots & 0 & 1 & m-1 \end{pmatrix},$$

where $m \in \mathbb{N}$. Then $\det(\mathbb{1}_n - D_m) = -2m + 1$, $\det(\mathbb{1}_n + D_m) = (-1)^{n-1}$ and $O(\mathbb{1}_n - D_m, 0) = 1$, therefore $R(\varphi_m) = m + 1$ and thus $\operatorname{Spec}_R(\Lambda) = \mathbb{N} \setminus \{1\} \cup \{\infty\}$. □

As we have now determined $\operatorname{Spec}_R(\mathbb{Z}^{n-k})$ and $\operatorname{Spec}_R(\Lambda_{k/k/0})$ for every k and n, we quickly deduce $\operatorname{Spec}_R(\Lambda_{n/k/0})$ with the help of lemma 2.5.18. The spectra can be found in table 7.1.

$n/k/\epsilon$	$\mathrm{Spec}_R(\Lambda)$
$1/1/0$	$\{\infty\}$
$2/1/0$	$\{\infty\}$
$2/2/0$	$2\mathbb{N} \cup \{3, \infty\}$
$3/1/0$	$\{\infty\}$
$3/2/0$	$4\mathbb{N} \cup \{6, \infty\}$
$3/3/0$	$\mathbb{N} \setminus \{1\} \cup \{\infty\}$
$4/1/0$	$\{\infty\}$
$4/2/0$	$2\mathbb{N} \cup 3\mathbb{N} \cup \{\infty\}$
$4/3/0$	$2\mathbb{N} \setminus \{2\} \cup \{\infty\}$
$4/4/0$	$\mathbb{N} \setminus \{1\} \cup \{\infty\}$
$n/1/0$	$\{\infty\}$
$n/2/0$	$2\mathbb{N} \cup 3\mathbb{N} \cup \{\infty\}$
$n/3/0$	$\mathbb{N} \setminus \{1\} \cup \{\infty\}$
$\vdots$	$\vdots$
$n/n-2/0$	$\mathbb{N} \setminus \{1\} \cup \{\infty\}$
$n/n-1/0$	$2\mathbb{N} \setminus \{2\} \cup \{\infty\}$
$n/n/0$	$\mathbb{N} \setminus \{1\} \cup \{\infty\}$

Table 7.1: Reidemeister spectra of the groups $\Lambda_{n/k/0}$

7.1.2 Bieberbach groups

Next, we consider the Bieberbach groups, i.e. exactly those groups $\Lambda_{n/k/\epsilon}$ with $\epsilon = 1$ and $1 \leq k \leq n-1$. We will start with the case $k = 1$.

Theorem 7.1.7. *The groups $\Lambda_{n/1/1}$ with $n \geq 2$ all have the R_∞-property.*

Proof. Let $n \geq 2$. From proposition 7.0.1 it follows that $\Lambda_{n/1/1}/Z(\Lambda_{n/1/1}) \cong \Lambda_{1/1/1} \cong \Lambda_{1/1/0}$. As shown in theorem 7.1.3, this (non-Bieberbach) quotient group has the R_∞-property, so by corollary 2.5.12 $\Lambda_{n/1/1}$ has the R_∞-property as well. □

Before we can discuss the groups $\Lambda_{n/k/1}$ with $k > 1$, we need the following lemma.

Lemma 7.1.8. *Let $D \in \mathbb{Z}^{n\times n}$. Then $|\det(\mathbb{1} - D)| + |\det(\mathbb{1} + D)| \in 2\mathbb{N}_0$.*

Proof. When projected element-wise to $\mathbb{Z}_2$, the matrices $\mathbb{1} - D$ and $\mathbb{1} + D$ are identical, hence their determinants have the same parity. As the absolute values have no influence on this parity, the sum of the absolute values of the determinants must be even. ☐

Theorem 7.1.9. *The groups* $\Lambda_{n/k/1}$ *with* $2 \leq k \leq n-1$ *have* $\mathrm{Spec}_R(\Lambda_{n/k/1}) = 2\mathbb{N} \cup \{\infty\}$.

Proof. For ease of notation, set $\Lambda := \Lambda_{n/k/1}$ and let $\varphi = \xi_{(d,D)} \in \mathrm{Aut}(\Lambda)$. Due to proposition 7.0.1, D must be of the form $D = \left(\begin{smallmatrix} D_1 & 0 \\ 0 & D_2 \end{smallmatrix}\right)$, where $D_1 \in \mathrm{GL}_k(\mathbb{Z})$ and $D_2 \in \mathrm{GL}_{n-k}(\mathbb{Z})$.

Setting $d = (d_1, d_2, \dots, d_n)$, one can calculate that

$$\varphi(\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_{n-k} \end{pmatrix}) = (\begin{pmatrix} 2d_1 \\ \vdots \\ 2d_k \\ D_2 \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1/2 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_{n-k} \end{pmatrix}).$$

Thus, $2d_i \in \mathbb{Z}$ for all $i = 1, \dots, k$ and there must exist integers $a_k, a_{k+1}, \dots, a_n$ such that D_2 is of the following form:

$$D_2 = \begin{pmatrix} * & \cdots & * & 2a_{k+1} \\ & & & 2a_{k+2} \\ \vdots & & \vdots & \vdots \\ & & & 2a_{n-1} \\ * & \cdots & * & 1+2a_n \end{pmatrix}. \tag{7.4}$$

Since $\Lambda_{n/k/1}$ is a Bieberbach group, we may apply theorem 4.2.6:

$$\begin{aligned} R(\varphi) &= \frac{1}{2}(\left|\det\left(\mathbb{1}_n - \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}\right)\right|_\infty + \left|\det\left(\mathbb{1}_n - \begin{pmatrix} -D_1 & 0 \\ 0 & D_2 \end{pmatrix}\right)\right|_\infty) \\ &= \frac{1}{2}\left|\det(\mathbb{1}_{n-k} - D_2)\right|_\infty \left(\left|\det(\mathbb{1}_k - D_1)\right|_\infty + \left|\det(\mathbb{1}_k + D_1)\right|_\infty\right). \end{aligned}$$

The last column of $\mathbb{1}_{n-k} - D_2$ is $(-2a_{k+1}, \dots, -2a_n)$ and therefore $|\det(\mathbb{1}_{n-k} - D_2)|_\infty \in 2\mathbb{N} \cup \{\infty\}$; from lemma 7.1.8 we know that the $|\det(\mathbb{1}_k - D_1)|_\infty + |\det(\mathbb{1}_k + D_1)|_\infty \in 2\mathbb{N} \cup \{\infty\}$. Combining this information, we obtain that $R(\varphi) \in 2\mathbb{N} \cup \{\infty\}$.

We now construct a family of automorphisms φ_m such that $R(\varphi_m) = 2m$ with $m \in \mathbb{N}$. Set $\varphi_m = \xi_{(0,D_m)}$ where D_{m1} and D_{m2} are chosen as follows:

- The matrix D_{m1}:
  - If $k=2$, take
  $$D_{m1} = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix}.$$
  - If $k \geq 3$, take
  $$D_{m1} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & \ddots & & & \vdots & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & 0 \\ \vdots & & \ddots & \ddots & 0 & m \\ 0 & \cdots & \cdots & 0 & 1 & m-1 \end{pmatrix}.$$

  In both cases $|\det(\mathbb{1}_k - D_{m1})| + |\det(\mathbb{1}_k + D_{m1})| = 2m$.
- The matrix D_{m2}:
  - If $n-k=1$, take $D_{m2} = -1$.
  - If $n-k=2$, take
  $$D_{m2} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$
  - If $n-k \geq 3$, take
  $$D_{m2} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \\ 1 & \ddots & & & \vdots & 0 & \vdots \\ 0 & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 & \vdots \\ \vdots & & & \ddots & 1 & -1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & -1 \end{pmatrix}.$$

  In all three cases $|\det(\mathbb{1} - D_{m2})| = 2$.

We have now found a family of automorphisms φ_m such that $R(\varphi_m) = 2m$ for every $m \in \mathbb{N}$. Hence $\mathrm{Spec}_R(\Lambda) = 2\mathbb{N} \cup \{\infty\}$. □

7.2 Reidemeister zeta functions

Since the rationality of Reidemeister zeta functions of (almost-)Bieberbach groups is known (see theorem 4.2.8), we can restrict ourselves to non-Bieberbach groups in this section. Most of the results presented in this section were published in [DTV18].

As shown in proposition 7.0.1, a non-torsion-free crystallographic group with diagonal holonomy $\mathbb{Z}_2$ is of the form

$$\Lambda_{n/k/0} = \Lambda_{k/k/0} \times \mathbb{Z}^{n-k},$$

with both factors characteristic. Because of example 2.6.7 and corollary 2.6.9, it suffices to find the Reidemeister zeta functions of $\Lambda_{k/k/0}$.

Let $\varphi = \xi_{(d/2,D)}$ be an automorphism of $\Lambda := \Lambda_{n/n/0}$, where $d \in \mathbb{Z}^n$ and $D \in \mathrm{GL}_n(\mathbb{Z})$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of D. We may then write, using proposition 7.1.6, that

$$\begin{aligned} R(\varphi) &= \frac{\left|\prod_{i=1}^n (1-\lambda_i)\right|_\infty + \left|\prod_{i=1}^n (1+\lambda_i)\right|_\infty}{2} + O(\mathbb{1}_n - D, d) \\ &= \frac{\prod_{i=1}^n |1-\lambda_i|_\infty + \prod_{i=1}^n |1+\lambda_i|_\infty}{2} + O(\mathbb{1}_n - D, d). \end{aligned}$$

Similarly, for any $k \in \mathbb{N}$ we have

$$R(\varphi^k) = \frac{\prod_{i=1}^n |1-\lambda_i^k|_\infty + \prod_{i=1}^n |1+\lambda_i^k|_\infty}{2} + O\left(\mathbb{1}_n - D^k, \left[\sum_{i=0}^{k-1} D^i\right] d\right).$$

We will deal with both terms separately. For the first term, we have the following lemma, which is very similar to what we did in example 2.6.7.

Lemma 7.2.1. *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of some matrix $D \in \mathrm{GL}_n(\mathbb{Z})$. Then there exist non-negative integers $a, b \in \mathbb{N}_0$ and complex numbers $\mu_1, \ldots, \mu_a, \nu_1, \ldots, \nu_b$ such that*

$$\frac{\prod_{i=1}^n |1-\lambda_i^k| + \prod_{i=1}^n |1+\lambda_i^k|}{2} = \mu_1^k + \mu_2^k + \cdots + \mu_a^k - \nu_1^k - \cdots - \nu_b^k$$

for each $k \in \mathbb{N}$.

Proof. We now consider 4 cases:

1. $\lambda_i \in \mathbb{R}$ and $|\lambda_i| < 1$. Then $|1-\lambda_i^k| = 1^k - \lambda_i^k$ and $|1+\lambda_i^k| = 1^k + \lambda_i^k$,

2. $\lambda_i \in \mathbb{R}$ and $\lambda_i < -1$. Then $|1 - \lambda_i^k| = -(-1)^k + (-\lambda_i)^k$ and $|1 + \lambda_i^k| = 1^k + \lambda_i^k$,

3. $\lambda_i \in \mathbb{R}$ and $\lambda_i > 1$. Then $|1 - \lambda_i^k| = -1^k + \lambda_i^k$ and $|1 + \lambda_i^k| = (-1)^k + (-\lambda_i)^k$.

4. $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$. Then its complex conjugate $\bar{\lambda}_i$ is an eigenvalue of D as well, and
$$|1 \pm \lambda_i^k||1 \pm \bar{\lambda}_i^k| = 1^k \pm \lambda_i^k \pm \bar{\lambda}_i^k + |\lambda_i|^{2k}.$$

Thus, we can expand both products $\prod_{i=1}^n |1 - \lambda_i^k|$ and $\prod_{i=1}^n |1 + \lambda_i^k|$ and obtain a sum of terms of the form $\pm\lambda_{i_1}^k \lambda_{i_2}^k \cdots \lambda_{i_p}^k = \pm(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_p})^k$ (where p varies between 0 and n). Note that all of these terms are, up to sign, k-th powers of terms which themselves do not depend on k. These two products will have exactly the same terms, though the sign of said terms may differ. If two matching terms have the same sign, their sum will have a factor 2 that cancels out with the 2 in the denominator; and if two matching terms have the opposite sign, they cancel out each other. So the entire term is indeed a sum and/or difference of k-th powers of fixed terms (not depending on k). $\square$

With this lemma proven, it is now easy to show the rationality of the first term.

Lemma 7.2.2. *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of some matrix $D \in \mathrm{GL}_n(\mathbb{Z})$. The function*
$$\exp \sum_{k=1}^{\infty} \frac{\prod_{i=1}^n |1 - \lambda_i^k| + \prod_{i=1}^n |1 + \lambda_i^k|}{2} \frac{z^k}{k}$$
is a rational function.

Proof. We invoke the previous lemma to obtain
$$\begin{aligned}
&\exp \sum_{k=1}^{\infty} \frac{\prod_{i=1}^n |1 - \lambda_i^k| + \prod_{i=1}^n |1 + \lambda_i^k|}{2} \frac{z^k}{k} \\
&= \exp \sum_{k=1}^{\infty} \frac{z^k}{k} \left(\sum_{i=1}^{a} \mu_i^k - \sum_{i=1}^{b} \nu_i^k \right) \\
&= \exp \left(\sum_{i=1}^{a} \sum_{k=1}^{\infty} \frac{\mu_i^k}{k} z^k - \sum_{i=1}^{b} \sum_{k=1}^{\infty} \frac{\nu_i^k}{k} z^k \right) \\
&= \exp \left(-\sum_{i=1}^{a} \log(1 - \mu_i z) + \sum_{i=1}^{b} \log(1 - \nu_i z) \right)
\end{aligned}$$

$$= \frac{\prod_{i=1}^{b}(1-\nu_i z)}{\prod_{i=1}^{a}(1-\mu_i z)},$$

which is a rational function. □

The second term is far less straightforward. We first introduce a particular family of sequences.

Definition 7.2.3. We define the sequence $a^i = (a^i_k)_{k\in\mathbb{N}}$ by

$$a^i_k = \begin{cases} i & \text{if } k \equiv 0 \bmod i, \\ 0 & \text{otherwise.} \end{cases}$$

The theorem below is essentially what we need to prove the rationality for the second term.

Theorem 7.2.4. *Let $D \in \mathrm{GL}_n(\mathbb{Z})$ and $d \in \mathbb{Z}^n$. Then there exist $l \in \mathbb{N}_0$ and $c_1, \ldots, c_l \in \mathbb{N}_0$ such that*

$$O\left(\mathbb{1}_n - D^k, \left[\sum_{i=0}^{k-1} D^i\right] d\right) = c_1 a^1_k + c_2 a^2_k + \cdots + c_l a^l_k$$

for all $k \in \mathbb{N}$.

Before we really start with the proof of this theorem, let us note that we do not need full information on the pair (d, D), but we only need to know their natural projections modulo 2, namely the pair $(\bar{d}, \bar{D})$, see lemma 7.1.4. To avoid having to write a bar above d and D each time we will assume from now onwards that $D \in \mathrm{GL}_n(\mathbb{Z}_2)$ and $d \in \mathbb{Z}_2^n$.

We will apply a change of base such that D has a more suitable form to work with. With that in mind, we first need the following matrix decomposition.

Lemma 7.2.5. *Let N be a nilpotent, upper-triangular $k \times k$-matrix and D an invertible $l \times l$-matrix over a field $\mathbb{F}$. For any $k \times l$-matrix B, there exists a (unique) $k \times l$-matrix X such that*

$$NX + XD = B. \tag{7.5}$$

Proof. We prove this by induction on k. If $k = 1$, then $N = 0$ and $X = BD^{-1}$. Now let $k \geq 2$ and suppose that the lemma holds for smaller values of k. Then N, X and B can be seen as block matrices of the forms

$$N = \left(\begin{array}{ccc|c} & N_1 & & N_2 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right), \quad X = \left(\begin{array}{c} X_1 \\ \hline X_2 \end{array}\right), \quad B = \left(\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right),$$

where N_1 is a nilpotent, upper-triangular $(k-1)\times(k-1)$-matrix, N_2 is a $(k-1)\times 1$-matrix, X_1 and B_1 are $(k-1)\times l$-matrices and X_2 and B_2 are $1\times l$-matrices. We can then split up (7.5) in the system of equations

$$\begin{cases} N_1X_2 + N_2X_2 + X_1D = B_1, \\ X_2D = B_2. \end{cases}$$

The second equation gives us $X_2 = B_2D^{-1}$, and substituting this into the first equation gives

$$N_1X_1 + X_1D = B_1 - N_2B_2D^{-1}.$$

By applying the induction hypothesis, we get a solution X_1. Together with X_2 we have the full solution X of (7.5). □

This decomposition allows us to put D in the required form.

Lemma 7.2.6. *Let D be an $n\times n$-matrix over a field $\mathbb{F}$, then there exists an invertible matrix P such that*

$$PDP^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

where D_1 is a unipotent, upper-triangular matrix, and D_2 does not have eigenvalue 1 (and hence $\mathbb{1} - D_2$ is invertible).

Proof. Consider the linear map

$$f : \mathbb{F}^n \to \mathbb{F}^n : \vec{x} \mapsto D\vec{x}.$$

It suffices to show that there exists a basis such that f has the required form with respect to this basis. Suppose that D has eigenvalue 1, then take an eigenvector corresponding to this eigenvalue and extend to a basis. With respect to this basis, we have

$$D \sim \left(\begin{array}{c|ccc} 1 & * & \cdots & * \\ \hline 0 & & & \\ 0 & & D' & \\ 0 & & & \end{array}\right).$$

We can then interpret D' as a linear map $\mathbb{F}^{n-1} \to \mathbb{F}^{n-1}$ and proceed by induction to obtain

$$D \sim \left(\begin{array}{c|c} D_1 & B \\ \hline 0 & D_2 \end{array}\right),$$

with D_1 a unipotent upper-triangular $k\times k$-matrix and D_2 an $l\times l$-matrix with no eigenvalue 1. Hence $D_1 - \mathbb{1}_k$ is a nilpotent upper-triangular $k\times k$-matrix and

$\mathbb{1}_l - D_2$ is an invertible $l \times l$-matrix. By lemma 7.2.5 there exists a $k \times l$-matrix X such that

$$(D_1 - \mathbb{1}_k)X + X(\mathbb{1}_l - D_2) = B,$$

which in turn gives

$$\begin{pmatrix} \mathbb{1}_k & X \\ 0 & \mathbb{1}_l \end{pmatrix} \begin{pmatrix} D_1 & B \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} \mathbb{1}_k & X \\ 0 & \mathbb{1}_l \end{pmatrix}^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

as required. □

From this point onwards, we will work with $\mathbb{F} = \mathbb{Z}_2$. To any pair (d, D), with $d \in \mathbb{Z}_2^n$ and $D \in \mathrm{GL}_n(\mathbb{Z}_2)$, we associate the sets V_k and W_k defined as

$$V_k = \left\{ x \in \mathbb{Z}_2^n \,\middle|\, (\mathbb{1} - D^k)x = \left[\sum_{i=0}^{k-1} D^i\right] d \right\},$$

$$W_k = \{x \in V_k \mid x \notin V_l \quad \forall l \in \{1, 2, \ldots, k-1\}\}.$$

Let $v_k = |V_k| = O\left(\mathbb{1}_n - D^k, \left[\sum_{i=0}^{k-1} D^i\right] d\right)$ and $w_k = |W_k|$. The W_k are disjoint sets and their union is all of $\mathbb{Z}_2^n$. Hence, it is obvious that only for a finite number of values of k we have that $w_k \neq 0$, since their sum equals 2^n. To prove theorem 7.2.4, we need to determine what the sequence $v = (v_k)_{k\in\mathbb{N}}$ is. As we have split up D in a unipotent block D_1 and a block with no eigenvalue 1, D_2, we will first restrict to these two blocks.

If D has no eigenvalue 1.

Let us first assume that D does not have eigenvalue 1, and therefore $\mathbb{1} - D$ is invertible. Then there exists some d_0 such that $(\mathbb{1} - D)d_0 = d$, and hence we can state

$$\left[\sum_{i=0}^{k-1} D^i\right] d = \left[\sum_{i=0}^{k-1} D^i\right] (\mathbb{1} - D)d_0 = (\mathbb{1} - D^k)d_0,$$

so we are actually searching for solutions of the linear system given by

$$(\mathbb{1} - D^k)(x - d_0) = 0.$$

The "shift" by d_0 has no effect on the number of solutions of this system, so we may assume without loss of generality that

$$V_k = \left\{x \in \mathbb{Z}_2^n \mid (\mathbb{1} - D^k)x = 0\right\}.$$

We will now formulate and prove some properties of these sets V_k and W_k.

Lemma 7.2.7. *Let $k < l$. If $x \in V_k \cap V_l$, then $x \in V_{l-k}$.*

Proof. Let $x \in V_k \cap V_l$. Then

$$\begin{aligned} 0 &= (\mathbb{1} - D^l)x \\ &= (\mathbb{1} - D^k + D^k - D^l)x \\ &= (\mathbb{1} - D^k)x + D^k(\mathbb{1} - D^{l-k})x \\ &= D^k(\mathbb{1} - D^{l-k})x, \end{aligned}$$

and because D is invertible we are left with $(\mathbb{1}-D^{l-k})x = 0$, hence $x \in V_{l-k}$. □

Corollary 7.2.8. *Let $k < l$. If $x \in V_k \cap V_l$, then*

(1) $x \in V_{l \bmod k}$,

(2) $x \in V_{\gcd(k,l)}$.

Proof. The first property follows by repeatedly applying lemma 7.2.7. The second property follows by repeatedly applying the first property. □

On the other hand, we also have

Lemma 7.2.9. *If $x \in V_k$, then $x \in V_{kl}$ for all $l \in \mathbb{N}$.*

Proof. All we have to do is split $(\mathbb{1} - D^{kl})$ in suitable factors:

$$(\mathbb{1} - D^{kl})x = (\mathbb{1} + D^k + \cdots + D^{(l-1)k})(\mathbb{1} - D^k)x = 0,$$

because $(\mathbb{1} - D^k)x = 0$. □

In conclusion, we can state that V_k is exactly the disjoint union

$$V_k = \bigsqcup_{d|k} W_d.$$

and hence we get

$$v_k = \sum_{d|k} w_d = \sum_{d|k} \frac{w_d}{d}(a^d)_k,$$

where we used that the k-th element in the sequence a^d is d, since k is a multiple of d. Now, since $(a^d)_k = 0$ when d does not divide k we have that

$$v_k = \sum_{d=1}^{k} \frac{w_d}{d}(a^d)_k = \sum_{d=1}^{\infty} \frac{w_d}{d}(a^d)_k.$$

So we indeed seem to have a sum of sequences a^d, but we still require the coefficients of this sum to be integers.

Lemma 7.2.10. *k divides w_k for any $k \in \mathbb{N}$.*

Proof. We define an action of $\mathbb{Z}$ on W_k by

$$\mathbb{Z} \times W_k \to W_k : (z, x) \mapsto z \cdot x = D^z x.$$

First, we verify that this action is well-defined. If $x \in W_k$, then

$$(\mathbb{1} - D^k)D^z x = D^z(\mathbb{1} - D^k)x = 0,$$

hence $D^z x \in V_k$. On the other hand, if for some $l < k$ we were to have that $D^z x \in V_l$, then

$$0 = (\mathbb{1} - D^l)D^z x = D^z(\mathbb{1} - D^l)x.$$

Because D is invertible, this would mean that $(\mathbb{1} - D^l)x = 0$, or in other words $x \in V_l$. This is a contradiction since $x \in W_k$. In fact, $k\mathbb{Z}$ acts trivially on W_k since

$$(\mathbb{1} - D^k)x = 0 \iff D^k x = x,$$

so we can redefine the original action as an action of $\mathbb{Z}_k$ on W_k, which is a free action. Indeed, suppose that for some $x \in W_k$ we have that $D^l x = x$, where l is not a multiple of k. Then $x \in V_l$ and therefore $x \in V_{l \bmod k}$. This obviously contradicts that $x \in W_k$.

By the orbit-stabiliser theorem, we can now partition W_k into finitely many orbits of length k, and thus k divides w_k. □

Putting everything together now, we can conclude that the sequence $v = (v_k)_{k\in\mathbb{N}}$ equals

$$v = \sum_{k=1}^{\infty} \frac{w_k}{k} a^k,$$

which has integer coefficients since k divides w_k. Recall that this is actually a finite sum, since only finitely many of the w_k are non-zero.

If D is unipotent upper-triangular.

For the case where D is unipotent upper-triangular, we will have very similar results as the previous case. The main difference here will be that we will end up working mainly with powers of 2 as opposed to arbitrary k. Because we are working over $\mathbb{Z}_2$, we have the following two statements:

Remark 7.2.11. If m is an odd positive integer, then for any integers $k_1, k_2, \dots, k_m$, we have that $D^{k_1} + D^{k_2} + \cdots + D^{k_m}$ is unipotent and upper-triangular (and hence invertible).

Remark 7.2.12. If D is a unipotent, upper-triangular, $n \times n$-matrix, then $D^{2^{n-1}} = \mathbb{1}_n$, since $\mathbb{1}_n - D^{2^{n-1}} = (\mathbb{1}_n - D)^{2^{n-1}} = 0$. This means that $V_{2^n} = \mathbb{Z}_2^n$.

The next lemma makes clear why we only really need to care about powers of 2.

Lemma 7.2.13. *Decompose k as $k = 2^r m$ with m odd. Then $V_k = V_{2^r}$.*

Proof. Let $M = \mathbb{1} + D^{2^r} + D^{2\cdot 2^r} + \cdots + D^{(m-1)2^r}$, which is invertible (see remark 7.2.11). Then

$$\begin{aligned}\mathbb{1} - D^k &= \mathbb{1} - D^{2^r m}\\ &= (\mathbb{1} + D^{2^r} + D^{2\cdot 2^r} + \cdots + D^{(m-1)2^r})(\mathbb{1} - D^{2^r})\\ &= M(\mathbb{1} - D^{2^r}),\end{aligned}$$

and

$$\left[\sum_{i=0}^{k-1} D^i\right] d = \left[\sum_{i=0}^{2^r m-1} D^i\right] d = M\left[\sum_{i=0}^{2^r-1} D^i\right] d.$$

We then obtain

$$\begin{aligned}(\mathbb{1} - D^k)x = \left[\sum_{i=0}^{k-1} D^i\right] d &\iff M(\mathbb{1} - D^{2^r})x = M\left[\sum_{i=0}^{2^r-1} D^i\right] d\\ &\iff (\mathbb{1} - D^{2^r})x = \left[\sum_{i=0}^{2^r-1} D^i\right] d,\end{aligned}$$

and therefore $V_k = V_{2^r}$. □

We conclude that $w_k = 0$ if k is not a power of 2. Now let r_0 be the smallest power of 2 such that $w_{2^{r_0}} \neq 0$, then there exists some d_0 such that

$$(\mathbb{1} - D^{2^{r_0}})d_0 = \left[\sum_{i=0}^{2^{r_0}-1} D^i\right] d.$$

Similarly to the other case, the number of elements $v_{2^{r_0}} = w_{2^{r_0}}$ in $V_{2^{r_0}} = W_{2^{r_0}}$ is equal to the number of solutions of the system

$$(\mathbb{1} - D^{2^{r_0}})(x - d_0) = 0.$$

Lemma 7.2.14. *2^{r_0} divides $w_{2^{r_0}}$.*

Proof. As we are working over $\mathbb{Z}_2$, we have that $\mathbb{1} - D^{2^{r_0}} = (\mathbb{1} - D)^{2^{r_0}}$. Since D is unipotent upper-triangular, $\mathbb{1} - D$ is nilpotent upper-triangular, and hence taking the 2^{r_0}-th power gives a matrix where the bottom r_0 rows are zero. Thus $w_{r_0} = |\ker(\mathbb{1} - D^{2^{r_0}})|$ is a multiple of 2^{r_0}. □

We have already shown that if $r = r_0$, we may work with the linear system $(\mathbb{1} - D^{2^r})(x - d_0) = 0$. This is, however, rather useless if we do not have this for every r. For $r > r_0$ we have

$$\begin{aligned}(\mathbb{1} - D^{2^r})x &= \left[\sum_{i=0}^{2^r-1} D^i\right] d \\ &= (\mathbb{1} + D^{2^{r_0}} + D^{2\cdot 2^{r_0}} + \cdots + D^{(2^{r-r_0}-1)2^{r_0}})\left[\sum_{i=0}^{2^{r_0}-1} D^i\right] d \\ &= (\mathbb{1} + D^{2^{r_0}} + D^{2\cdot 2^{r_0}} + \cdots + D^{(2^{r-r_0}-1)2^{r_0}})(\mathbb{1} - D^{2^{r_0}})d_0 \\ &= (\mathbb{1} - D^{2^r})d_0.\end{aligned}$$

So indeed we end up with the linear system

$$(\mathbb{1} - D^{2^r})(x - d_0) = 0,$$

and again we may assume without loss of generality that $d_0 = 0$. Now that we have this system for all $r \geq r_0$, we also want to generalise lemma 7.2.14 to all $r > r_0$.

Lemma 7.2.15. *2^r divides w_{2^r} for all $r > r_0$.*

Proof. Analogously to lemma 7.2.10, we have an action of $\mathbb{Z}_{2^r}$ on W_{2^r}. Suppose this action is not free. As subgroups of $\mathbb{Z}_{2^r}$ are generated by divisors of 2^r, there then exist some $x \in W_{2^r}$ and some $r' < r$ such that $D^{2^{r'}}x = x$, which contradicts that $x \in W_{2^r}$. So 2^r divides w_{2^r}. □

The following steps are identical to the case where D has no eigenvalue 1, hence we leave these to the reader and we can conclude that also in this case

$$v = \sum_{k=1}^{\infty} \frac{w_k}{k} a^k,$$

where $\frac{w_k}{k}$ is an integer and the sum is in fact finite.

The general case

We now have all the necessary tools to prove theorem 7.2.4.

Proof of theorem 7.2.4. Earlier in this subsection we proved that, after a change of basis, D is a block matrix of the form

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

such that D_1 is unipotent upper-triangular and D_2 has no eigenvalue 1. We may split the vector d in two pieces d_1 and d_2 matching the sizes of D_1 and D_2 respectively. So for any k, we have two linear systems of equations given by

$$\begin{cases} (\mathbb{1} - D_1^k)x_1 = \left[\sum_{i=0}^{k-1} D_1^i\right] d_1, \\ (\mathbb{1} - D_2^k)x_2 = \left[\sum_{i=0}^{k-1} D_2^i\right] d_2. \end{cases}$$

The total number of solutions x is of course the number of pairs (x_1, x_2). In the previous subsections we have shown that both "subsystems" give sequences

$$v = (v_1, v_2, v_3, \dots),$$

$$v' = (v'_1, v'_2, v'_3, \dots),$$

that are linear combinations of the sequences a^i, say $v = \sum_{k=1}^{n} c_k a^k$ and $v' = \sum_{l=1}^{m} c'_l a^l$. To solve the linear system as a whole, we are actually looking for the sequence $v \cdot v'$ given by the component-wise multiplication of v and v':

$$v \cdot v' = (v_1 v'_1, v_2 v'_2, v_3 v'_3, \dots).$$

Using that $a^k \cdot a^l = \gcd(k,l) a^{\operatorname{lcm}(k,l)}$, we get

$$v \cdot v' = \left(\sum_{k=1}^{n} c_k a^k\right)\left(\sum_{l=1}^{m} c'_l a^l\right) = \sum_{k,l} c_k c'_l \gcd(k,l) a^{\operatorname{lcm}(k,l)},$$

and since $c_k c'_l \gcd(k,l)$ is a non-negative integer for all k and l, this proves the theorem. □

Corollary 7.2.16. *Let $D \in \mathrm{GL}_n(\mathbb{Z})$ and $d \in \mathbb{Z}^n$. The function*

$$\exp \sum_{k=1}^{\infty} O\left(\mathbb{1}_n - D^k, \left[\sum_{i=0}^{k-1} D^i\right] d\right) \frac{z^k}{k}$$

is a rational function.

Proof. From theorem 7.2.4 we know that

$$O\left(\mathbb{1}_n - D^k, \left[\sum_{i=0}^{k-1} D^i\right] d\right) = \sum_{i=1}^{l} c_i a_k^i,$$

for certain $c_1, \ldots, c_l$. Hence:

$$\begin{aligned}\exp \sum_{k=1}^{\infty} O\left(\mathbb{1}_n - D^k, \left[\sum_{i=0}^{k-1} D^i\right] d\right) \frac{z^k}{k} &= \exp \sum_{k=1}^{\infty} \left[\sum_{i=1}^{l} c_i a_k^i\right] \frac{z^k}{k} \\ &= \exp \sum_{i=1}^{l} c_i \left[\sum_{k=1}^{\infty} a_k^i \frac{z^k}{k}\right] \\ &= \prod_{i=1}^{l} \exp\left[-c_i \log(1 - z^i)\right] \\ &= \prod_{i=1}^{l} (1 - z^i)^{-c_i},\end{aligned}$$

which is a rational function. □

With both terms taken care of, we can now state the following theorem:

Theorem 7.2.17. *Let φ be an automorphism of the group $\Lambda = \langle \mathbb{Z}^n, (0, -\mathbb{1}_n)\rangle$ such that $R(\varphi) < \infty$. Then there exist $a, b, l \in \mathbb{N}_0$, $\mu_1, \ldots, \mu_a, \nu_1, \ldots, \nu_b \in \mathbb{C}$ and $c_1, \ldots, c_l \in \mathbb{N}_0$ such that*

$$R_z(\varphi) = \frac{\prod_{i=1}^{b}(1 - \nu_i z)}{\prod_{i=1}^{a}(1 - \mu_i z) \prod_{i=1}^{l}(1 - z^i)^{c_i}}.$$

The radius of convergence r of this function is given by

$$r = \frac{1}{\max\{1, |\mu_1|, \ldots, |\mu_a|, |\nu_1|, \ldots, |\nu_b|\}}$$

if at least some $c_i \neq 0$, otherwise it is given by

$$r = \frac{1}{\max\{|\mu_1|, \dots, |\mu_a|, |\nu_1|, \dots, |\nu_b|\}}.$$

Finally, we may then conclude that all Reidemeister zeta functions of non-torsion-free crystallographic groups with diagonal holonomy $\mathbb{Z}_2$ are rational.

Theorem 7.2.18. *Let $\varphi = \varphi_1 \times \varphi_2$ be an automorphism of a non-torsion-free crystallographic group $\Lambda_{n/k/0} \cong \Lambda_{k/k/0} \times \mathbb{Z}^{n-k}$ with diagonal holonomy $\mathbb{Z}_2$, such that $R(\varphi) < \infty$. Then $R_\varphi(z) = R_{\varphi_1}(z) * R_{\varphi_2}(z)$, the convolution of $R_{\varphi_1}(z)$ and $R_{\varphi_2}(z)$, is a rational function.*

Combining this with theorem 4.2.8 gives finally gives us the following.

Corollary 7.2.19. *Let φ be an automorphism of a crystallographic group with diagonal holonomy $\mathbb{Z}_2$. If $R_\varphi(z)$ exists, it is a rational function.*

While we have now determined the rationality, we have not yet proven that these Reidemeister zeta functions actually exist.

Theorem 7.2.20. *Let $\Lambda = \Lambda_{n/k/\epsilon}$ be a crystallographic group with diagonal holonomy $\mathbb{Z}_2$. Then it admits Reidemeister zeta functions of automorphisms if and only if $k \geq 2$ and $n - k \neq 1$.*

Proof. Let $\varphi = \xi_{(d,D)}$ be an automorphism, then D must necessarily be of the form

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2, \end{pmatrix},$$

with $D_1 \in \mathrm{GL}_k(\mathbb{Z})$, $D_2 \in \mathrm{GL}_{n-k}(\mathbb{Z})$. Moreover, if Λ is a Bieberbach group, then D_2 must be of the form (7.4) as well.

First, consider the case $k = 1$. Then by theorem 7.1.3 Λ has the R_∞-property, and therefore does not admit Reidemeister zeta functions of automorphisms.

Second, consider the case $n - k = 1$. Then D_2 is either 1 or -1, so either way D_2^2 equals 1. But then

$$\det(\mathbb{1}_n - D^2) = \det(\mathbb{1}_{n-1} - D_1^2)\det(1 - D_2^2) = 0,$$

hence $R(\varphi^2) = \infty$ and thus $R_\varphi(z)$ does not exist.

Finally, assume that $n - k \neq 1$ and $k \geq 2$. Let $M_2 \in \mathrm{GL}_2(\mathbb{Z})$ and $M_3 \in \mathrm{GL}_3(\mathbb{Z})$ be the matrices

$$M_2 := \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad M_3 := \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and note that all of their eigenvalues λ satisfy $|\lambda| \neq 1$. Depending on whether k and $n-k$ are even or odd, take D_1 and D_2 as

$$D_i = \begin{pmatrix} M_2 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_2 \end{pmatrix} \text{ or } D_i = \begin{pmatrix} M_2 & & & & \\ & M_2 & & & \\ & & \ddots & & \\ & & & M_2 & \\ & & & & M_3 \end{pmatrix}.$$

In the case $n-k=0$, there is no D_1 and we simply have that $D = D_2$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of D, which will satisfy $|\lambda_i^k| \neq 1$ for all $k \in \mathbb{N}$, and thus

$$\det(\mathbb{1}_n \pm D^k) = \prod_{i=1}^{n} |1 \pm \lambda_i^k| \neq 0.$$

By theorem 4.2.5 we then find that $R(\varphi^k) < \infty$. Next, we study the radius of convergence. Let λ_{max} be the eigenvalue with the largest modulus, which definitely satisfies $|\lambda_{max}| > 1$. From proposition 2.5.14 we know that

$$\begin{aligned} R(\varphi^k) &\leq |\det(\mathbb{1}_n - D)| + |\det(\mathbb{1}_n + D)| \\ &= \prod_{i=1}^{n} |1 - \lambda_i^k| + \prod_{i=1}^{n} |1 + \lambda_i^k| \\ &\leq 2 \prod_{i=1}^{n} \left(|\lambda_i|^k + 1\right) \\ &\leq 2 \left(|\lambda_{max}|^k + 1\right)^n \\ &\leq 2 \left(2|\lambda_{max}|^k\right)^n \\ &= 2^{n+1} |\lambda_{max}|^{nk}, \end{aligned}$$

hence for $k \geq 2^{n+1}$ we have that

$$\frac{R(\varphi^k)}{k} \leq \frac{2^{n+1}|\lambda_{max}|^{nk}}{k} \leq |\lambda_{max}|^{nk}.$$

Thus, the radius of converge r satisfies

$$r^{-1} = \limsup_{k\to\infty} \sqrt[k]{\frac{R(\varphi^k)}{k}} \leq |\lambda_{max}|^n,$$

hence $r \geq |\lambda_{max}|^{-n} > 0$ and therefore $R_\varphi(z)$ exists. □

We conclude this section with an example of a Reidemeister zeta function on $\Lambda_{2/2/0}$.

Example 7.2.21. Let $\Gamma = \langle \mathbb{Z}^2, (0, -\mathbb{1}_2)\rangle$ and let $F \in \mathrm{GL}_2(\mathbb{Z})$ be the matrix

$$F := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

which is also known as the *Fibonacci matrix*, as its powers generate the Fibonacci sequence:

$$F^n = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1}, \end{pmatrix}$$

where F_k is the k-th Fibonacci number, i.e.

- $F_0 = 0$,
- $F_1 = 1$,
- $F_{k+1} = F_k + F_{k-1}$.

Consider the automorphism $\varphi = \xi_{(d/2,F)}$ for any $d \in \mathbb{Z}^2$. One can calculate that

$$|\det(\mathbb{1}_2 \pm F^k)| = \phi^k + (1-\phi)^k \pm (1 + (-1)^k),$$

with ϕ the golden ratio, and

$$O\left(\mathbb{1}_2 - F^k, \left[\sum_{i=0}^{k-1} F\right] d\right) = \begin{cases} 4 & \text{if } k \equiv 0 \bmod 3, \\ 1 & \text{if } k \equiv 1, 2 \bmod 3. \end{cases}$$

Thus,

$$\begin{aligned} R_\varphi(z) &= \exp \sum_{k=1}^{\infty} \frac{R(\varphi^k)}{k} z^k \\ &= \exp \sum_{k=1}^{\infty} \frac{\phi^k + (1-\phi)^k + a_k^1 + a_k^3}{k} z^k \\ &= \frac{1}{(1-\phi z)(1-(1-\phi)z)(1-z)(1-z^3)} \\ &= \frac{1}{(1-z-z^2)(1-z)(1-z^3)}, \end{aligned}$$

which is indeed a rational function and has radius of convergence $1/\phi$.

Chapter 8

Generalised Hantzsche-Wendt groups

This chapter is largely based on [DDP09] and extends the results of that paper.

8.1 Definitions and properties

Definition 8.1.1. A square $n \times n$-matrix $(a_{ij})_{ij}$ is called *circulant* if $a_{ij} = a_{i+1\,j+1}$ for all $1 \leq i, j \leq n$, where the indices are taken modulo n when necessary. In other words, the $j+1$-th row of the matrix is the j-th row shifted one position to the right.

Definition 8.1.2. Let $\sigma \in \mathcal{S}_n$ be a permutation and $k_1, \ldots, k_n \in \mathbb{R}$. Define

$$M_\sigma(k_1, \ldots, k_n) = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix},$$

where

$$m_{ij} = \begin{cases} k_j & \text{if } i = \sigma(j) \\ 0 & \text{else} \end{cases}.$$

The following proposition tells us what an automorphism of a GHW group must look like.

Proposition 8.1.3 (see [DDP09, Proposition 5.6])**.** *Let Γ be a HW group in standard form. Let $\varphi = \xi_{(d,D)}$ be an automorphism of Γ. Then there exist $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ and a permutation $\sigma \in \mathcal{S}_n$ such that $D = M_\sigma(\epsilon_1, \dots, \epsilon_n)$.*

The proposition below is, in some sense, a converse to the previous proposition.

Proposition 8.1.4 (see [DDP09, Proposition 5.7])**.** *Let Γ be a HW group in standard form with associated matrix $(a_{ij})_{ij}$. Let $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ and $\sigma \in \mathcal{S}_n$ such that $a_{ij} = a_{\sigma(i)\sigma(j)}$ for all $1 \leq i, j \leq n$. Then there exists an automorphism $\varphi = \xi_{(d,D)}$ of Γ with $D = M_\sigma(\epsilon_1, \dots, \epsilon_n)$.*

8.2 The Reidemeister spectra and zeta functions

The GHW groups that have the R_∞-property have been determined by Dekimpe, De Rock and Penninckx in [DDP09].

Theorem 8.2.1 (see [DDP09, Theorem 5.9])**.** *A non-orientable GHW group has the R_∞-property. A HW group does not have the R_∞-property if and only if it is isomorphic to a HW group in standard form whose associated matrix is circulant.*

An important part of the proof of this theorem can be summarised in the following proposition.

Proposition 8.2.2. *Let Γ be a HW group in standard form with associated matrix $A = (a_{ij})_{ij}$. Let $\varphi = \xi_{(d,D)}$ be an automorphism of Γ with $D = M_\sigma(\epsilon_1, \dots, \epsilon_n)$. If σ is not a cycle of full length, then $R(\varphi) = \infty$.*

We will expand on theorem 8.2.1 by explicitly calculating the Reidemeister spectrum for those HW groups that do not have the R_∞-property. To do so, we will need to slightly generalise the following lemma.

Lemma 8.2.3 (see [DDP09, Lemma 5.5])**.** *Let $\sigma \in \mathcal{S}_n$ be the permutation $(1\ 2\ \cdots\ n)$ and let $k_1, \dots, k_n \in \mathbb{R}$. Then*

$$\det(\mathbb{1}_n - M_\sigma(k_1, \dots, k_n)) = 1 - k_1 \cdots k_n.$$

Corollary 8.2.4. *The above lemma holds for any cycle $\sigma \in \mathcal{S}_n$ of full length.*

Proof. Since all cycles of the same length are conjugate, there exists a permutation τ such that $\tau^{-1} \circ \sigma \circ \tau$ is exactly the cycle $(1\ 2\ \cdots\ n)$. Let $P := M_\tau(1, \dots, 1)$, then

$$P^{-1} M_\sigma(k_1, \dots, k_n) P = M_{(1\ 2\ \dots\ n)}(k_{\tau(1)}, \dots, k_{\tau(n)}).$$

The result then follows from applying lemma 8.2.3. □

Theorem 8.2.5. *Let Γ be a HW group in standard form whose associated matrix is circulant. Then its Reidemeister spectrum is $\{2, \infty\}$.*

Proof. First, let us prove that any automorphism φ with finite Reidemeister number must have $R(\varphi) = 2$. Let $\varphi = \xi_{(d,D)}$ be an automorphism of Γ with finite Reidemeister number. By proposition 8.2.2, $D = M_\sigma(\epsilon_1, \ldots, \epsilon_n)$ with σ a cycle of full length.

Let $A \in F$, which is a diagonal matrix with an even number of -1's on its diagonal. Then

$$AD = M_\sigma(\epsilon'_1, \ldots, \epsilon'_n) \text{ with } \epsilon'_1\epsilon'_2 \cdots \epsilon'_n = \epsilon_1\epsilon_2 \cdots \epsilon_n.$$

By corollary 8.2.4 we have that

$$\det(\mathbb{1}_n - AD) = 1 - \epsilon'_1\epsilon'_2 \cdots \epsilon'_n = 1 - \epsilon_1\epsilon_2 \cdots \epsilon_n.$$

Theorem 4.2.5 then implies that, since $R(\varphi) < \infty$, we must have $\epsilon_1\epsilon_2 \cdots \epsilon_n = -1$ such that in turn

$$\det(\mathbb{1}_n - AD) = 1 - \epsilon_1\epsilon_2 \cdots \epsilon_n = 2,$$

for every $A \in F$. Applying the averaging formula (theorem 4.2.6) we find that $R(\varphi) = 2$.

Conversely, we would like to show that Γ must admit such automorphism. Let $\sigma = (1\ 2\ \cdots\ n)$, which satisfies $a_{ij} = a_{\sigma(i)\sigma(j)}$ for all $1 \leq i, j \leq n$. By proposition 8.1.4, Γ then admits an automorphism $\varphi = \xi_{(d,D)}$ with $D = M_\sigma(-1, 1, \ldots, 1)$. □

We end this section with the following result on Reidemeister zeta functions.

Theorem 8.2.6. *A GHW group does not admit Reidemeister zeta functions.*

Proof. It suffices to prove that for any automorphism φ of a GHW group Γ, there exists some $k \in \mathbb{N}$ such that $R(\varphi^k) = \infty$. By theorem 8.2.1, the only non-trivial case is when Γ is a HW group in standard form with circulant associated matrix. As shown in theorem 8.2.5, the only automorphisms with finite Reidemeister number are those of the form $\varphi = \xi_{(d,D)}$ with $D = M_\sigma(\epsilon_1, \ldots, \epsilon_n)$, where σ is a cycle of full length and $\epsilon_1\epsilon_2 \cdots \epsilon_n = -1$.

Now, consider φ^{2n} with n the dimension of Γ. Since σ is a cycle of length n, D^n is a diagonal matrix whose diagonal entries are either -1 or 1, and thus $D^{2n} = \mathbb{1}_n$. Since $\varphi^{2n} = \xi_{(d', D^{2n})}$ for some $d' \in \mathbb{R}^n$, it is easy to check using theorem 4.2.5 that $R(\varphi^{2n}) = \infty$. □

8.3 Uniqueness of HW groups without the R_∞-property

Miatello and Rossetti show in [MR99b] that for each odd dimension $n > 1$, there exists a HW group Γ in standard form with circulant associated matrix. In [DDP09], the authors conjecture that in every odd dimension n, this HW group is unique (up to isomorphism), and they verified this conjecture for all odd $n \leq 21$.

Conjecture 8.3.1 (see [DDP09, Conjecture 5.11])**.** In every odd dimension $n > 1$, there is (up to isomorphism) a unique HW group that does not have the R_∞-property, and its associated matrix $(a_{ij})_{ij}$ satisfies

$$a_{ij} = \begin{cases} 1/2 & \text{if } j - i \equiv 0, 1 \bmod n, \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

A general idea of solving this conjecture is the following. Consider the set of all circulant $n \times n$-matrices whose entries are either 0 or 1/2. For every matrix in this set, check using proposition 3.3.19 if it corresponds to a HW group Γ. If it does, the conjecture above implies that Γ is isomorphic to the HW group defined by (8.1).

The proposition below shows that we can more or less halve the number of circulant matrices we need to check.

Proposition 8.3.2. *Let Γ be an n-dimensional HW group in standard form with circulant associated matrix. Then Γ is isomorphic to a HW group in standard form with circulant associated matrix, for which every column has at most $\frac{n+1}{2}$ non-zero entries.*

Proof. The associated matrix $(a_{ij})_{ij}$ of Γ is completely determined by any of its columns a_i, and every column has the same number of non-zero entries. If a_i has at most $\frac{n+1}{2}$ non-zero entries, there is nothing to prove. So assume that a_i has more than $\frac{n+1}{2}$ non-zero entries, and recall from proposition 3.3.19 that necessarily $a_{ii} = 1/2$. Let $d = (1/4, 1/4, \ldots, 1/4)$ and consider the inner automorphism $\iota_{(d, -\mathbb{1}_n)} : \mathrm{Aff}(\mathbb{R}^n) \to \mathrm{Aff}(\mathbb{R}^n)$. We have that

$$(d, -\mathbb{1}_n)(a_i, A_i)(d, -\mathbb{1}_n)^{-1} = ((\mathbb{1}_n - A_i)d - a_i, A_i),$$

hence if we set $a_i' := (\mathbb{1}_n - A_i)d - a_i + e_i$, we find

$$a_{ij}' = \begin{cases} 1/2 & \text{if } i = j, \\ 1/2 & \text{if } i \neq j, a_{ij} = 0, \\ 0 & \text{if } i \neq j, a_{ij} = 1/2. \end{cases}$$

In particular, each a'_i has less than $\frac{n+1}{2}$ non-zero entries. Then $\Gamma' := \iota_{(d,-\mathbb{1}_n)}(\Gamma)$ is a HW group in standard form with circulant associated matrix $(a'_{ij})_{ij}$, and is isomorphic to Γ. □

A circulant matrix is completely determined by its first column, hence proposition 3.3.19 can be restated in terms of this first column. Using our knowledge of HW groups and the proposition above, we can state the following conjecture, which implies conjecture 8.3.1.

Conjecture 8.3.3. Let $k \in \mathbb{N}$ and $x \in \{0,1\}^{2k+1}$, and label the components of x by $x_0, x_1, \dots, x_{2k}$. Assume that $x_0 = 1$ and $\#\{i \in \{1, 2, \dots, 2k\} \mid x_i = 1\}$ is odd and at most k. If for every $I \subsetneq \{0, 1, 2, \dots, 2k\}$ with $\#I$ odd:

$$\exists j \in I : \#\{i \in I \mid x_{i-j} = 1\} \text{ is odd,} \tag{8.2}$$

where the index $i - j$ is taken modulo $2k + 1$ when necessary, then exactly two components are 1, i.e. there is a unique $i \in \{1, \dots, 2k\}$ such that $x_i = 1$.

We verified conjecture 8.3.3 for all $k \leq 13$, which implies conjecture 8.3.1 holds for all odd $n \leq 27$.

Remark 8.3.4. In the conjecture above, it is not necessary to check every set I:

- If $\#I = 1$, then the condition always holds.
- If condition (8.2) holds for some set I, then it also holds for $I + 1 := \{i + 1 \bmod 2k + 1 \mid i \in I\}$, $I + 2$, ... , $I + 2k$.

Remark 8.3.5. The converse of conjecture 8.3.3 is not true. Consider the tuple $x = (1, 0, 0, 1, 0, 0, 0, 0, 0)$ and the set $I = \{0, 3, 6\}$. We have that

$$\forall j \in I : \#\{i \in I \mid x_{i-j} = 1\} = 2.$$

Chapter 9

The R_∞-property

The 1-, 2- and 3-dimensional almost-crystallographic groups that do not have the R_∞-property were determined by Dekimpe and Penninckx in [DP11, Section 4]. We extend these results to the 4-dimensional groups, and in the case of crystallographic groups with finite outer automorphism group even up to the 6-dimensional groups.

9.1 Crystallographic groups

In this section, we determine which crystallographic groups have (or do not have) the R_∞-property. We do this for all crystallographic groups up to dimension 4, and for the crystallographic groups with finite outer automorphism group up to dimension 6. The results obtained in this section were published in [DKT19].

Multiple classification systems for crystallographic groups exist (especially in dimensions 2 and 3). An important part of these classifications are the concepts of $\mathbb{Q}$-classes and $\mathbb{Z}$-classes.

Definition 9.1.1. Two n-dimensional crystallographic groups are said to belong to the same $\mathbb{Q}$-class ($\mathbb{Z}$-class) if their holonomy groups are conjugate in $\mathrm{GL}_n(\mathbb{Q})$ ($\mathrm{GL}_n(\mathbb{Z})$).

In particular, a $\mathbb{Q}$-class consists of one or multiple $\mathbb{Z}$-classes. After conjugation, we may assume that every crystallographic group Γ in a fixed $\mathbb{Z}$-class has the exact same holonomy group $F \subseteq \mathrm{GL}_n(\mathbb{Z})$, which we will also call the *holonomy* *group of the $\mathbb{Z}$-class.* We can use this to create canonical generating sets for crystallographic groups.

Let F be the holonomy group of a $\mathbb{Z}$-class, and enumerate the elements of F by $A_1, \ldots, A_k$. Now let Γ be any crystallographic group in the $\mathbb{Z}$-class. For every A_i, pick the unique element $a_i \in \mathbb{R}^n$ with $0 \leq a_i < 1$ such that $(a_i, A_i) \in \Gamma$. We define

$$F_{ext}(\Gamma) := \{(a_1, A_1), \ldots, (a_k, A_k)\},$$

hence $\Gamma = \langle \mathbb{Z}^n, F_{ext}(\Gamma) \rangle$. In particular, the $\mathbb{Z}$-class contains the crystallographic group $\mathbb{Z}^n \rtimes F$, for which

$$F_{ext}(\mathbb{Z}^n \rtimes F) = \{(0, A_1), \ldots, (0, A_k)\}.$$

This group is very useful for determining the R_∞-property of groups in the $\mathbb{Z}$-class, as illustrated by the theorem below.

Theorem 9.1.2. *Let F be the holonomy group of an n-dimensional $\mathbb{Z}$-class of crystallographic groups. If $\mathbb{Z}^n \rtimes F$ has the R_∞-property, then so does every other crystallographic group in the same $\mathbb{Z}$-class.*

Proof. Consider a crystallographic group Γ belonging to the same $\mathbb{Z}$-class as $\mathbb{Z}^n \rtimes F$ and let $\varphi = \xi_{(d,D)} \in \mathrm{Aut}(\Gamma)$. Then $D \in N_F = N_{\mathrm{GL}_n(\mathbb{Z})}(F)$ and thus $\varphi' = \xi_{(0,D)}$ is an automorphism of $\mathbb{Z}^n \rtimes F$. Since $\mathbb{Z}^n \rtimes F$ has the R_∞-property we find that $R(\varphi') = \infty$, and by theorem 4.2.5 we obtain

$$R(\varphi') = \infty \iff \exists A \in F : \det(\mathbb{1}_n - AD) = 0 \iff R(\varphi) = \infty,$$

hence Γ has the R_∞-property as well. □

The converse is not necessarily true. For a crystallographic group Γ with holonomy group F, the projection

$$p : \mathrm{Aut}(\Gamma) \to N_F : \xi_{(d,D)} \mapsto D$$

does not have to be surjective. In other words, given a matrix $D \in N_F$, there may not exist a $d \in \mathbb{R}^n$ such that the map $\xi_{(d,D)} : \gamma \mapsto (d, D)\gamma(d, D)^{-1}$ is an automorphism of Γ. We define N_Γ as the image of the projection p above, which is therefore a subgroup of N_F.

Algorithm 1 provides a method to check whether or not a matrix $D \in N_F$ is actually an element of N_Γ. Moreover, if $D \in N_\Gamma$, it calculates an explicit $d \in \mathbb{Q}^n$ such that $\xi_{(d,D)} \in \mathrm{Aut}(\Gamma)$. The (more general) idea behind this algorithm is described in [Lut13, Section 4.1].

Algorithm 1 Determining if $D \in N_\Gamma$

function ExtendsToAutomorphism(D, Γ)
  $\sigma \leftarrow$ Permutation in $\mathcal{S}_{\#F}$ for which $A_{\sigma(i)} = DA_iD^{-1}$
  $M \leftarrow \begin{pmatrix} \mathbb{1}_n - A_{\sigma(1)} \\ \mathbb{1}_n - A_{\sigma(2)} \\ \vdots \\ \mathbb{1}_n - A_{\sigma(k)} \end{pmatrix}$
  $m \leftarrow \begin{pmatrix} Da_1 - a_{\sigma(1)} \\ Da_2 - a_{\sigma(2)} \\ \vdots \\ Da_k - a_{\sigma(k)} \end{pmatrix}$
  $P, Q, S \leftarrow$ matrices such that $PMQ = S$, the Smith normal form of M
  $t \leftarrow Pm$
  $r \leftarrow \text{rank}(S)$
  if $t_{r+1}, \ldots, t_{nk} \in \mathbb{Z}$ **then**
    for $i \in \{1, \ldots, r\}$ **do**
      $d'_i \leftarrow -t_i / S_{i,i}$
    end for
    for $i \in \{r+1, \ldots, n\}$ **do**
      $d'_i \leftarrow 0$
    end for
    $d \leftarrow Qd'$
    return d
  else
    return `fail`
  end if
end function

Theorem 9.1.3. *Let* Γ *be an* n*-dimensional crystallographic group with holonomy group* F*. Given a matrix* $D \in N_F$*,* `ExtendsToAutomorphism`(D, Γ) *returns* `fail` *if* $D \notin N_\Gamma$*, or returns a* $d \in \mathbb{Q}^n$ *such that* $\xi_{(d,D)} \in \text{Aut}(\Gamma)$ *if* $D \in N_\Gamma$*.*

Proof. The map $F \to F : A \mapsto DAD^{-1}$ is an automorphism of F since $D \in N_F$. If $F_{ext}(\Gamma) = \{(a_1, A_1), \ldots, (a_k, A_k)\}$, we can associate a permutation $\sigma \in \mathcal{S}_{\#F}$ to this map such that $A_{\sigma(i)}$ is the image of A_i.

If there exists a $d \in \mathbb{R}^n$ such that $\xi_{(d,D)} \in \operatorname{Aut}(\Gamma)$, then it must satisfy $(d, D)(a_i, A_i)(d, D)^{-1} \in \Gamma$ for every $i = 1, \dots, k$. Equivalently,

$$\begin{aligned} &(d, D)(a_i, A_i)(d, D)^{-1} \in \Gamma \\ &\iff (d, D)(a_i, A_i)(d, D)^{-1}(a_{\sigma(i)}, A_{\sigma(i)})^{-1} \in \mathbb{Z}^n \\ &\iff Da_i - a_{\sigma(i)} + (\mathbb{1}_n - A_{\sigma(i)})d \in \mathbb{Z}^n. \end{aligned} \tag{9.1}$$

Therefore, construct the matrices

$$M := \begin{pmatrix} \mathbb{1}_n - A_{\sigma(1)} \\ \mathbb{1}_n - A_{\sigma(2)} \\ \vdots \\ \mathbb{1}_n - A_{\sigma(k)} \end{pmatrix} \in \mathbb{Z}^{nk \times n}, \quad m := \begin{pmatrix} Da_1 - a_{\sigma(1)} \\ Da_2 - a_{\sigma(2)} \\ \vdots \\ Da_k - a_{\sigma(k)} \end{pmatrix} \in \mathbb{Z}^{nk},$$

and calculate the matrices $P \in \operatorname{GL}_{nk}(\mathbb{Z})$, $S \in \mathbb{Z}^{nk \times n}$ and $Q \in \operatorname{GL}_n(\mathbb{Z})$ such that S is the Smith normal form of M and $PMQ = S$. With these matrices known, calculate $t := Pm$ and define $d' := Q^{-1}d$, and observe that condition (9.1) is equivalent to

$$t + Sd' \in \mathbb{Z}^{nk}. \tag{9.2}$$

Let r be the rank of the matrix S and let $s_1, s_2, \dots, s_r$ be the (non-zero) invariant factors of S. Writing out the components of $t + Sd'$, we find that condition (9.2) means that $t_i + s_i d'_i \in \mathbb{Z}$ for $i = 1, \dots, r$ and $t_i \in \mathbb{Z}$ for $i = r + 1, \dots, nk$.

The latter condition does not depend on d. Thus, we verify if $t_{r+1}, \dots, t_{nk} \in \mathbb{Z}$. If this is not the case, the required d does not exist and the algorithm returns "`fail`". Otherwise, we set $d'_i = -t_i/s_i$ for $i = 1, \dots, r$ and $d'_i = 0$ for $i = r + 1, \dots, n$, and calculate $d = Qd'$, which will be an element of $\mathbb{Q}^n$. The map $\xi_{(d,D)} : \Gamma \to \Gamma : \gamma \mapsto (d, D)\gamma(d, D)^{-1}$ is then an automorphism of Γ. □

Remark 9.1.4. It is not necessary to use every (a_i, A_i) in $F_{ext}(\Gamma)$ in algorithm 1. If $i_1, \dots, i_r$ are indices such that $A_{i_1}, \dots, A_{i_r}$ is a generating set of the holonomy group F, we can construct M, m as

$$M := \begin{pmatrix} \mathbb{1}_n - A_{\sigma(i_1)} \\ \vdots \\ \mathbb{1}_n - A_{\sigma(i_r)} \end{pmatrix} \in \mathbb{Z}^{nr \times n}, \quad m := \begin{pmatrix} Da_{i_1} - a_{\sigma(i_1)} \\ \vdots \\ Da_{i_r} - a_{\sigma(i_r)} \end{pmatrix} \in \mathbb{Z}^{nr},$$

where r may be much smaller than $k = \#F$. This can significantly speed up computations.

If a crystallographic group Γ has finite outer automorphism group, or equivalently N_F is finite, we can use algorithm 2 to determine whether Γ has the R_∞-property

Algorithm 2 Determining if a crystallographic group Γ has the R_∞-property

```
function HasRinfinityProperty(Γ)
    N_F ← N_{GL_n(ℤ)}(F)
    if #N_F = ∞ then
        return fail                                          ▷ # Out(Γ) = ∞
    else
        N_Γ ← ∅                                              ▷ Calculate N_Γ
        for D ∈ N_F do
            if ExtendsToAutomorphism(D, Γ) ≠ fail then
                N_Γ ← N_Γ ∪ {D}
            end if
        end for
        for D ∈ N_Γ do                          ▷ Find D with R(ξ_(d,D)) < ∞
            R_∞ ← false
            for A ∈ F do
                if det(𝟙_n − AD) = 0 then
                    R_∞ ← true
                end if
            end for
            if R_∞ = false then
                return false
            end if
        end for
        return true
    end if
end function
```

or not. This is basically the algorithm from [DP11] combined with algorithm 1, meaning no work has to be done by hand anymore.

However, in its presented form, algorithm 2 is not very efficient. Algorithm 3 is an extended version of algorithm 2 and takes an entire $\mathbb{Z}$-class as input, rather than a single crystallographic group. It outputs the list of groups in this $\mathbb{Z}$-class that do not have the R_∞-property. Running algorithm 3 for a $\mathbb{Z}$-class is significantly faster than running algorithm 2 separately for every crystallographic group in this $\mathbb{Z}$-class. Remark that further improvements to this algorithm can be made, using the following facts:

- If for some $D \in N_F$ we have that $\det(\mathbb{1}_n - AD) \neq 0$ for all $A \in F$, then the same holds for any matrix in the same coset of N_F/F as D. This also follows from lemma 2.5.19(1).

- If for some $D \in N_F$ we have that $\det(\mathbb{1}_n - AD) \neq 0$ for all $A \in F$, then the same holds for any matrix in the same conjugacy class of N_F as D. This also follows from lemma 2.5.19(2).
- If for some $D \in N_\Gamma$, there exists a d such that $\xi_{(d,D)} \in \operatorname{Aut}(\Gamma)$, then the same holds for any matrix in the same coset of N_F/F as D. This follows from taking the compositions of the form $\iota_{(a,A)} \circ \xi_{(d,D)}$.

Algorithm 3 Determining which crystallographic groups in a $\mathbb{Z}$-class Z do not have the R_∞-property

```
function HasRinfinityPropertyZClass(Z)
    N_F ← N_{GL_n(ℤ)}(F)
    if #N_F = ∞ then
        return fail                                   ▷ # Out(Γ) = ∞
    else
        D_{<∞} ← ∅              ▷ Find D with R(ξ_{(d,D)}) < ∞ whenever d exists
        for D ∈ N_F do
            R_∞ ← false
            for A ∈ F do
                if det(𝟙_n − AD) = 0 then
                    R_∞ ← true
                end if
            end for
            if R_∞ = false then
                D_{<∞} ← D_{<∞} ∪ {D}
            end if
        end for
        Γ_{<∞} ← ∅                           ▷ Find Γ without R_∞-property
        for D ∈ D_{<∞} do
            for Γ ∈ Z do
                if ExtendsToAutomorphism(D, Γ) ≠ fail then
                    Γ_{<∞} ← Γ_{<∞} ∪ {Γ}
                end if
            end for
        end for
        return Γ_{<∞}
    end if
end function
```

However, these algorithms fail when the outer automorphism group is infinite (and hence N_F is infinite), in which case we can try two things:

1. Show that all crystallographic groups in a $\mathbb{Z}$-class with holonomy group F have the R_∞-property, by finding a characteristic subgroup N of $\mathbb{Z}^n \rtimes F$ such that $(\mathbb{Z}^n \rtimes F)/N$ has the R_∞-property. This relies on corollary 2.5.12 and theorem 9.1.2.

2. Show that a crystallographic group Γ does not have the R_∞-property, by checking for random matrices $D \in N_F$ whether they belong to N_Γ (using algorithm 1) and whether any automorphism with D as linear part can have finite Reidemeister number (using theorem 4.2.5).

However, unlike the aforementioned algorithms, these are trial-and-error methods that have to be done manually.

We have applied algorithm 3 for all crystallographic groups up to dimension 6. Up to dimension 4, we also applied the methods mentioned above for the crystallographic groups with infinite outer automorphism group, and for every $\mathbb{Z}$-class we either found that all groups have the R_∞-property, or we found an automorphism $\varphi = \xi_{(d,D)}$ with $R(\varphi) < \infty$ for every group in the $\mathbb{Z}$-class. Therefore, we have completely determined which crystallographic groups up to dimension 4 have the R_∞-property.

To create a library of crystallographic groups and calculate the normalisers N_F, we used CARAT [Car06]. Our algorithms were implemented in GAP [GAP18], and we used the GAP-package `carat` [Car18] to access the aforementioned library.

For the crystallographic groups with finite outer automorphism group up to dimension 6, the results of algorithm 3 can be found in tables B.1 to B.6. For the groups with infinite outer automorphism group up to dimension 4, tables B.7 to B.9 provide pairs (d, D) for the groups that do not have the R_∞-property, and tables B.10 and B.11 provide quotient groups of $\mathbb{Z}^n \rtimes F$ for the $\mathbb{Z}$-classes that do have the R_∞-property. We summarise these results in table 9.1.

In these tables, we identify the groups using 3 different classification systems. Up to dimension 3, there is the classification system from the International Tables for Crystallography [Aro16], where groups are identified by n/IT, with n the dimension and IT the specific group. Up to dimension 4, there is the notation from [Bro+78], where groups are identified by $n/c/q/z/s$, with n the dimension, c the crystal system, q the $\mathbb{Q}$-class, z the $\mathbb{Z}$-class and s the specific group. In every dimension, there is the CARAT-notation [Car06], where groups are identified by q-z-s, with q the $\mathbb{Q}$-class, z the $\mathbb{Z}$-class and s the specific group.

dim	# groups	(# Out(Γ) < ∞)	no R_∞	(# Out(Γ) < ∞)
1	2	(2)	1	(1)
2	17	(15)	2	(1)
3	219	(204)	12	(7)
4	4783	(4 388)	91	(45)
5	222 018	(204 768)	?	(146)
6	28 927 915	(26 975 265)	?	(321)

Table 9.1: Crystallographic groups up to dimension 6 without the R_∞-property

9.2 Almost-crystallographic groups

In this section we determine which almost-crystallographic groups up to dimension 4 do not have the R_∞-property. We first give two propositions, which restrict which almost-crystallographic groups and automorphisms respectively can have finite Reidemeister numbers. The results obtained in this section were published in [Ter19].

Proposition 9.2.1. *Let Γ be an almost-crystallographic group with translation subgroup N of rank $n \geq 3$ and nilpotency class $c \geq 2$ with $\sqrt[N]{\gamma_c(N)} \cong \mathbb{Z}$. If the holonomy group F acts non-trivially on $\sqrt[N]{\gamma_c(N)}$, then Γ has the R_∞-property.*

Proof. Let $A \in F$ arbitrary, $\varphi = \xi_{(d,D)} \in \operatorname{Aut}(\Gamma)$ and $x \in N$ such that $\langle x \rangle = \sqrt[N]{\gamma_c(N)}$. Since A acts on x by ${}^A x = x^{\epsilon_A}$ with $\epsilon_A \in \{-1, 1\}$ and $\varphi(x) = x^\nu$ with $\nu \in \{-1, 1\}$, then (after a change of basis) A_* and D_* must have the following forms:

$$A_* = \begin{pmatrix} \epsilon_A & * & \cdots & * \\ 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}, \quad D_* = \begin{pmatrix} \nu & * & \cdots & * \\ 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}.$$

Thus, $\mathbb{1}_n - A_* D_*$ is of the form

$$\mathbb{1}_n - A_* D_* = \begin{pmatrix} 1 - \nu\epsilon_A & * & \cdots & * \\ 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}.$$

Now let us look at specific $A \in F$. First, let A be the neutral element of F, which necessarily acts trivially on x. The above matrix then has upper left entry $1 - \nu$, hence $\det(\mathbb{1}_n - D_*) \neq 0$ if and only if $\nu = -1$.

Second, let A be an element of F for which $\epsilon_A = -1$. Such element exists since we assumed F acts non-trivially on $\sqrt[N]{\gamma_c(N)}$. Then the matrix $\mathbb{1}_n - A_* D_*$ has upper left entry $1+\nu$, and $\det(\mathbb{1}_n - A_* D_*) \neq 0$ if and only if $\nu = 1$.

As ν cannot be -1 and 1 at the same time, we always have some $A \in F$ for which $\det(\mathbb{1}_n - A_* D_*) = 0$, and by theorem 4.2.5 this means that $R(\varphi) = \infty$. Since this holds for any automorphism, Γ has the R_∞-property. □

Proposition 9.2.2. *Let Γ be an almost-crystallographic group with translation subgroup N of rank $n \geq 3$ and nilpotency class $c \geq 2$, such that $\sqrt[N]{\gamma_c(N)} \cong \mathbb{Z}$. If the restriction of $\varphi \in \mathrm{Aut}(\Gamma)$ to $\sqrt[N]{\gamma_c(N)}$ is the identity, then $R(\varphi) = \infty$.*

Proof. Let $\varphi = \xi_{(d,D)} \in \mathrm{Aut}(\Gamma)$ and $x \in N$ such that $\langle x \rangle = \sqrt[N]{\gamma_c(N)}$. If $\varphi(x) = x$, then (after a change of basis) D_* has the form

$$D_* = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix},$$

and hence $\det(\mathbb{1}_n - D_*) = 0$. By theorem 4.2.5 this means that $R(\varphi) = \infty$. □

9.2.1 Dimension 3

In this case the translation subgroup N is a finitely generated, torsion-free, nilpotent group of rank 3 and nilpotency class at most 2. Nilpotency class 1 is of course the crystallographic case, which was done in the previous section, so let Γ be an almost-crystallographic group whose translation subgroup N is a nilpotent group of rank 3 and nilpotency class 2. This group N can be given the following presentation:

$$\left\langle e_1, e_2, e_3 \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & [e_3, e_2] = e_1^{l_1} \\ [e_3, e_1] = 1 & \end{array} \right\rangle.$$

Moreover, let G be the Lie group that Γ is modelled on. By [Dek95, Theorem 4.1], there exists a faithful affine representation $\lambda : G \rtimes \mathrm{Aut}(G) \to \mathrm{Aff}(\mathbb{R}^3)$ such that its restriction to Γ is again a faithful affine representation. In particular,

$$\lambda(e_1) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \lambda(e_2) = \begin{pmatrix} 1 & 0 & -\frac{l_1}{2} & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \lambda(e_3) = \begin{pmatrix} 1 & \frac{l_1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the value of l_1 is determined by the relation $[e_3, e_2] = e_1^{l_1}$.

Corollary 3.4.9 tells us that the subgroup $\langle e_1 \rangle = \sqrt[N]{\gamma_2(N)}$ is characteristic and the quotient $\Gamma' := \Gamma/\langle e_1 \rangle$ is a 2-dimensional crystallographic group. Using corollary 2.5.15, we know that if Γ' has the R_∞-property, then so does Γ. In [DE02; Dek96] the almost-crystallographic groups were classified into families based on which crystallographic group Γ' is. Since only three 2-dimensional crystallographic groups do not have the R_∞-property (min.2-1.1-0, group.1-1.1-0 and min.5-1.1-0) we need only consider the corresponding three families of 3-dimensional almost-crystallographic groups. We will name these three families after the quotient group Γ'.

Family min.2-1.1-0. This family consists of the finitely generated, torsion-free, nilpotent groups of nilpotency class 2 and rank 3. We have already determined in section 5.2.1 that these groups do not have the R_∞-property.

Family min.5-1.1-0. Every group in this family has a presentation of the form

$$\left\langle e_1, e_2, e_3, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_1^{k_2} e_3 \alpha \\ [e_3, e_2] = e_1^{k_1} & \alpha e_3 = e_1^{k_3} e_2^{-1} e_3^{-1} \alpha \\ & \alpha^3 = e_1^{k_4} \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & k_2 & -\frac{k_1}{2} + k_3 & \frac{k_4}{3} \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the translation subgroup N and the isolator $\sqrt[N]{\gamma_2(N)} = \langle e_1 \rangle$ are characteristic, any automorphism $\varphi = \xi_{(d,D)}$ must be of the form

$$\varphi(e_1) = e_1^{\det(M)},$$

$$\varphi(e_2) = e_1^{n_1} e_2^{m_1} e_3^{m_2},$$

$$\varphi(e_3) = e_1^{n_2} e_2^{m_3} e_3^{m_4},$$

$$\varphi(\alpha) = e_1^{n_3} e_2^{n_4} e_3^{n_5} \alpha^{\epsilon},$$

where

$$M = \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \quad n_i \in \mathbb{Z}, \quad \epsilon \in \{-1, 1\}.$$

Then D_* is of the form

$$D_* = \begin{pmatrix} \det(M) & * & * \\ 0 & m_1 & m_3 \\ 0 & m_2 & m_4 \end{pmatrix}.$$

Let $A \in F$, and let A' be the projection of A to F', the holonomy group of $\Gamma' := \Gamma/\langle e_1 \rangle$ (this is of course the crystallographic group min.5-1.1-0). Then

$$\det(\mathbb{1}_3 - A_* D_*) = (1 - \det(M)) \det(\mathbb{1}_2 - A'M).$$

We may calculate that (using algorithm 1) that $\#N_{\Gamma'} = 12$, and $N_{\Gamma'}$ is exactly the set of possible matrices M. Six of the matrices M in $N_{\Gamma'}$ have determinant 1, in which case $\det(\mathbb{1}_3 - A_* D_*) = 0$ for all $A \in F$. Thus, using theorem 4.2.5, we can see that these automorphisms have infinite Reidemeister number. For the other six matrices M, there always exists some $A' \in F'$ such that $\det(\mathbb{1}_2 - A'M) = 0$. Again, using theorem 4.2.5, these automorphisms have infinite Reidemeister number. This result was also obtained in [DP11, Theorem 4.4].

Family group.1-1.1-0. Every group in this family has a presentation of the form

$$\left\langle e_1, e_2, e_3, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_1^{k_2} e_2^{-1} \alpha \\ [e_3, e_2] = e_1^{k_1} & \alpha e_3 = e_1^{k_3} e_3^{-1} \alpha \\ & \alpha^2 = e_1^{k_4} \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & k_2 & k_3 & \frac{k_4}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define an automorphism $\varphi = \xi_{(d,D)}$ by

$$\varphi(e_1) = e_1^{-1},$$

$$\varphi(e_2) = e_1^{k_1 - k_2 - k_3} e_2 e_3^2,$$

$$\varphi(e_3) = e_1^{3k_1 - k_2 - 2k_3} e_2^2 e_3^3,$$

$$\varphi(\alpha) = e_1^{-k_4} \alpha,$$

then D_* is of the form
$$D_* = \begin{pmatrix} -1 & * & * \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}.$$
We can apply theorem 4.2.5 to show that $R(\varphi) < \infty$ and hence Γ does not have the R_∞-property. This result was also obtained in [DP11, Theorem 4.4].

9.2.2 Dimension 4

In this case the translation subgroup N is a finitely generated, torsion-free, nilpotent group of rank 4 and nilpotency class at most 3. Nilpotency class 1 is of course the crystallographic case, which was done in the previous section.

Nilpotency class 2

Let Γ be an almost-crystallographic group whose translation subgroup N is a nilpotent group of rank 4 and nilpotency class 2. The group N can be given the following presentation:
$$\left\langle e_1, e_2, e_3, e_4 \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & [e_3, e_2] = e_1^{l_1} \\ [e_3, e_1] = 1 & [e_4, e_2] = e_1^{l_2} \\ [e_4, e_1] = 1 & [e_4, e_3] = e_1^{l_3} \end{array} \right\rangle.$$
Moreover, let G be the Lie group that Γ is modelled on. By [Dek95, Theorem 4.1], there exists a faithful affine representation $\lambda : G \rtimes \mathrm{Aut}(G) \to \mathrm{Aff}(\mathbb{R}^4)$ such that its restriction to Γ is again a faithful affine representation. In particular,
$$\lambda(e_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \lambda(e_2) = \begin{pmatrix} 1 & 0 & -\frac{l_1}{2} & -\frac{l_2}{2} & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\lambda(e_3) = \begin{pmatrix} 1 & \frac{l_1}{2} & 0 & -\frac{l_3}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \lambda(e_4) = \begin{pmatrix} 1 & \frac{l_2}{2} & \frac{l_3}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
where the values of l_1, l_2 and l_3 are determined by the relations $[e_3, e_2] = e_1^{l_1}$, $[e_4, e_2] = e_1^{l_2}$ and $[e_4, e_3] = e_1^{l_3}$.

Again, the subgroup $\langle e_1 \rangle = \sqrt[N]{\gamma_2(N)}$ is characteristic and the quotient $\Gamma' := \Gamma/\langle e_1 \rangle$ is a 3-dimensional crystallographic group. Just like the three-dimensional case, we need only consider the families whose quotient Γ' does not have the R_∞-property. As calculated in the previous section, there are twelve such families. These families can be split in smaller subfamilies, determined by the action of F on $\sqrt[N]{\gamma_2(N)}$: every $A \in F$ acts on e_1 by ${}^A e_1 = e_1^{\epsilon_A}$ with $\epsilon_A \in \{-1, 1\}$. By proposition 9.2.1 we need only consider those subfamilies where F acts trivially on $\sqrt[N]{\gamma_2(N)}$.

Remark 9.2.3. In the three-dimensional case, the action of F on $\sqrt[N]{\gamma_2(N)}$ was always uniquely determined. If the crystallographic quotient group $\Gamma' := \Gamma/\langle e_1 \rangle$ has holonomy group F', then the action of $A \in F$ on e_1 is given by ${}^A e_1 = e_1^{\det(A')}$, with A' the projection of A to F'.

In the classification of the 4-dimensional almost-crystallographic groups in [Dek96], it turned out (using techniques from [Dek96, Section 5.4]) that for an almost-crystallographic group belonging to one of the families min.10-1.1-0, min.10-1.1-3, min.10-1.3-0, min.10-1.4-0 or min.10-1.4-1, F acting trivially on $\sqrt[N]{\gamma_2(N)}$ implies that the group is actually crystallographic. Therefore we may omit these families and we are left with only 7 families to study.

Note that the presentations given below may vary from those in [DE02; Dek96]. Let Γ_1 and λ_1 denote a group and its faithful representation as given below, and let Γ_2 and λ_2 be the corresponding group and representation as given by [Dek96] or [DE02]. Table B.12 contains a matrix δ such that

$$\lambda_1(\Gamma_1) = \delta \lambda_2(\Gamma_2) \delta^{-1},$$

hence $\lambda_1(\Gamma_1)$ and $\lambda_2(\Gamma_2)$ are conjugate subgroups of $\mathrm{Aff}(\mathbb{R}^4)$ and therefore Γ_1 and Γ_2 are isomorphic.

Family min.6-1.1-0. This family consists of the finitely generated, torsion-free, nilpotent groups of nilpotency class 2 and rank 4. We have already determined in section 5.2.2 that these groups do not have the R_∞-property.

Families min.7-1.1-0, min.7-1.1-1 and min.7-1.2-0. Every group in one of these families has a presentation of the form

$$\left\langle e_1, e_2, e_3, e_4, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ {[e_3, e_1]} = 1 & \alpha e_2 = e_2 \alpha \\ {[e_4, e_1]} = 1 & \alpha e_3 = e_1^{k_2} e_2^{-\nu} e_3^{-1} \alpha \\ {[e_3, e_2]} = 1 & \alpha e_4 = e_1^{k_3} e_4^{-1} \alpha \\ {[e_4, e_2]} = 1 & \alpha^2 = e_1^{k_4} e_2^{\mu} \\ {[e_4, e_3]} = e_1^{k_1} & \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & 0 & k_2 & k_3 & \frac{k_4}{2} \\ 0 & 1 & -\nu & 0 & \frac{\mu}{2} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Family min.7-1.1-0 is given by $\mu, \nu = 0$, family min.7-1.1-1 by $\mu = 1, \nu = 0$ and family min.7-1.2-0 by $\mu = 0, \nu = 1$. Define an automorphism $\varphi = \xi_{(d,D)}$ by

$$\varphi(e_1) = e_1^{-1},$$

$$\varphi(e_2) = e_2^{-1},$$

$$\varphi(e_3) = e_1^{k_1-k_2-k_3} e_2^{\nu} e_3 e_4^2,$$

$$\varphi(e_4) = e_1^{3k_1-k_2-2k_3} e_2^{\nu} e_3^2 e_4^3,$$

$$\varphi(\alpha) = e_1^{-k_4} e_2^{-\mu} \alpha,$$

then D_* is of the form

$$D_* = \begin{pmatrix} -1 & * & * & * \\ 0 & -1 & * & * \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{pmatrix}.$$

We can apply theorem 4.2.5 to show that $R(\varphi) < \infty$ and hence Γ does not have the R_∞-property.

Families min.13-1.1-0 and min.13-1.2-0. Every group in one of these families has a presentation of the form

$$\left\langle e_1, e_2, e_3, e_4, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_2 \alpha \\ [e_4, e_1] = 1 & \alpha e_3 = e_1^{k_2} e_4 \alpha \\ [e_3, e_2] = 1 & \alpha e_4 = e_1^{k_3} e_2^{\mu} e_3^{-1} e_4^{-1} \alpha \\ [e_4, e_2] = 1 & \alpha^3 = e_1^{k_4} \\ [e_4, e_3] = e_1^{k_1} & \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & 0 & k_2 & -\frac{k_1}{2} + k_3 & \frac{k_4}{3} \\ 0 & 1 & 0 & \mu & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Family min.13-1.1-0 is given by by $\mu = 1$ and family min.13-1.2-0 by $\mu = 0$. Since the translation subgroup N, the centre $Z(N) = \langle e_1, e_2 \rangle$ and the isolator $\sqrt[N]{\gamma_2(N)} = \langle e_1 \rangle$ are characteristic, any automorphism $\varphi = \xi_{(d,D)}$ must be of the form

$$\begin{aligned} \varphi(e_1) &= e_1^{\det(M)}, \\ \varphi(e_2) &= e_1^{n_1} e_2^{\tau} \\ \varphi(e_3) &= e_1^{n_2} e_2^{n_3} e_3^{m_1} e_4^{m_2}, \\ \varphi(e_4) &= e_1^{n_4} e_2^{n_5} e_3^{m_3} e_4^{m_4}, \\ \varphi(\alpha) &= e_1^{n_6} e_2^{n_7} e_3^{n_8} e_4^{n_9} \alpha^{\epsilon}, \end{aligned}$$

where

$$M = \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \quad n_i \in \mathbb{Z}, \quad \epsilon, \tau \in \{-1, 1\}.$$

Then D_* is of the form

$$D_* = \begin{pmatrix} \det(M) & * & * & * \\ 0 & \tau & * & * \\ 0 & 0 & m_1 & m_3 \\ 0 & 0 & m_2 & m_4 \end{pmatrix}.$$

Let $A \in F$, and let A' be the projection of A to F', the holonomy group of $\Gamma' := \Gamma/Z(N)$ (this is the crystallographic group min.5-1.1-0). Then

$$\det(\mathbb{1}_4 - A_* D_*) = (1 - \det(M))(1 - \tau) \det(\mathbb{1}_2 - A'M).$$

Just like in section 9.2.1, family min.5-1.1-0, there are only 12 possible matrices M, and for each of them either $\det(M) = 1$ or there exists some $A' \in F'$ such that $\det(\mathbb{1}_2 - A'M) = 0$. By theorem 4.2.5, these automorphisms have infinite Reidemeister number.

Family group.5-1.1-0. Every group in this family has a presentation of the form

$$\left\langle e_1, e_2, e_3, e_4, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_1^{k_4} e_2^{-1} \alpha \\ [e_4, e_1] = 1 & \alpha e_3 = e_1^{k_5} e_3^{-1} \alpha \\ [e_3, e_2] = e_1^{k_1} & \alpha e_4 = e_1^{k_6} e_4^{-1} \alpha \\ [e_4, e_2] = e_1^{k_2} & \alpha^2 = e_1^{k_7} \\ [e_4, e_3] = e_1^{k_3} & \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & k_4 & k_5 & k_6 & \frac{k_7}{2} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Set $k := \gcd(k_1, k_2, k_3)$ and $g := e_2^{k_3/k} e_3^{-k_2/k} e_4^{k_1/k}$, then the centre $Z(N)$ of the translation subgroup is generated by e_1 and g. Let $\varphi : \Gamma \to \Gamma$ be any automorphism. Since $\langle e_1 \rangle$ and $Z(N)$ are both characteristic in Γ, we have that $\varphi(g) = g^\epsilon e_1^m$ for some $\epsilon \in \{-1, 1\}$ and $m \in \mathbb{Z}$. Consider the induced automorphism $\varphi' = \xi_{(d', D')}$ on $\Gamma' := \Gamma / \langle e_1 \rangle$, the crystallographic group group.5-1.1-0. Then

$$\varphi'(g\langle e_1 \rangle) = D'(g\langle e_1 \rangle) = \varphi(g)\langle e_1 \rangle = g^\epsilon \langle e_1 \rangle.$$

Depending on the value of ϵ, D'_* has either eigenvalue 1, in which case $\det(\mathbb{1}_3 - D'_*) = 0$, or eigenvalue -1, in which case $\det(\mathbb{1}_3 + D'_*) = 0$. Since the holonomy group of Γ' is $\{\mathbb{1}_3, -\mathbb{1}_3\}$, we obtain by theorem 4.2.5 that $R(\varphi') = \infty$ and by lemma 2.5.10 that therefore $R(\varphi) = \infty$. Since this holds for an arbitrary automorphism, Γ has the R_∞-property.

Nilpotency class 3

In section 5.2.2, we have determined that the finitely generated, torsion-free, nilpotent groups of nilpotency class 3 and rank 4 have the R_∞-property. Applying corollary 2.5.12 then proves that every 4-dimensional almost-crystallographic group with translation subgroup of nilpotency class 3 has the R_∞-property.

Chapter 10

Reidemeister spectra

10.1 Crystallographic groups

In this section, we will calculate the Reidemeister spectra of all crystallographic groups of dimension at most 3 that do not have the R_∞-property, with partial results up to dimension 6. The results obtained in this section were published in [DKT19] and [DTV19].

For crystallographic groups with finite outer automorphism group, we will present algorithms that calculate the Reidemeister spectrum. For the remaining groups, we will have to proceed by hand, which is feasible up to dimension 3. For dimension 4, we limit ourselves to calculating the spectra of only a small number of groups, mainly those where we have some extra tools to help us (for example lemma 2.5.18 or theorem 4.2.6).

10.1.1 Finite outer automorphism group

To calculate the Reidemeister spectrum of a crystallographic group Γ with finite outer automorphism group, we need two main algorithms: a first one to calculate the Reidemeister number of a given automorphism $\varphi = \xi_{(d,D)}$, and a second one to construct a set of representatives of $\mathrm{Out}(\Gamma)$.

Calculating the Reidemeister number of an automorphism

We will start by explaining the approach to calculating Reidemeister numbers in a more general setting. From proposition 2.5.13 and its proof, we obtain the following theorem.

Theorem 10.1.1. *Let G be a group with normal subgroup $N \lhd G$, and let φ be an endomorphism such that $\varphi(N) \subseteq N$. Denote by φ' the induced endomorphism on G/N. Then the set of Reidemeister classes of φ is given by*

$$\mathfrak{R}(\varphi) = \bigsqcup_{[gN]_{\varphi'} \in \mathfrak{R}(\varphi')} (\psi_g \circ \hat{\iota}_g)(\mathfrak{R}(\iota_g \varphi|_N)),$$

where ψ_g is the following bijective map:

$$\psi_g : \hat{p}_g^{-1}([1N]_{\iota_{gN}\varphi'}) \to \hat{p}^{-1}([gN]_{\varphi'}) : [h]_{\iota_g\varphi} \mapsto [hg]_{\varphi}.$$

The composite map $(\psi_g \circ \hat{\iota}_g)$ in the above theorem is, in general, not injective. We have that

$$\begin{aligned} &(\psi_g \circ \hat{\iota}_g)([n_1]_{\iota_g\varphi|_N}) = (\psi_g \circ \hat{\iota}_g)([n_2]_{\iota_g\varphi|_N}) \\ \iff\ & \hat{\iota}_g([n_1]_{\iota_g\varphi|_N}) = \hat{\iota}_g([n_2]_{\iota_g\varphi|_N}) \\ \iff\ & \exists h \in G : n_1 = hn_2(\iota_g\varphi)(h)^{-1} \\ \iff\ & \exists h \in G : n_1 g = hn_2 g\varphi(h)^{-1} \\ \iff\ & n_1 g \sim_\varphi n_2 g. \end{aligned}$$

Thus, if we have a group G with a normal subgroup N and an endomorphism φ as described in theorem 10.1.1, and we are able to do the following:

- calculating a set of representatives of the Reidemeister classes $\mathfrak{R}(\varphi')$ of G/N,
- calculating a set of representatives of the Reidemeister classes $\mathfrak{R}(\iota_g\varphi|_N)$ of N,
- checking if two elements of G are φ-twisted equivalent,

then we can use theorem 10.1.1 to calculate a set of representatives of the Reidemeister classes $\mathfrak{R}(\varphi)$ of G.

Let us now apply this to the setting of crystallographic groups. Let $\varphi = \xi_{(d,D)}$ be an automorphism of a crystallographic group Γ. We will take $N = \mathbb{Z}^n$, the translation subgroup of Γ, and then $G/N = F$, the holonomy group. Since F is finite, it is not hard to calculate the Reidemeister classes of the induced automorphism φ' on this group. For example, algorithm 4 is a simple algorithm to calculate a set of representatives.

Algorithm 4 Calculating representatives of $\mathfrak{R}(\varphi)$ for a finite group G

```
function CalculateRepresentativesFiniteGroup(G, φ)
    R ← ∅
    for g ∈ G do
        new ← true
        for (g', h) ∈ R × G do
            if g = hg'φ(h)^{-1} then
                new ← false                    ▷ g ∼_φ g' for some g' ∈ R
            end if
        end for
        if new = true then
            R ← R ∪ {g}                        ▷ g represents a new Reidemeister class
        end if
    end for
    return R
end function
```

To calculate a set representatives of the Reidemeister classes in $\mathbb{Z}^n$, note that we have shown in examples 2.5.8 and 2.5.9 that for any $D \in \mathbb{Z}^{n\times n}$,

$$\mathfrak{R}(D) = \mathbb{Z}^n / \operatorname{im}(\mathbb{1}_n - D).$$

Thus, it suffices to find a representative of every coset of $\operatorname{im}(\mathbb{1}_n - D)$ in $\mathbb{Z}^n$. This can be done with algorithm 5, as long as the determinant of $\mathbb{1}_n - D$ is non-zero. We prove the correctness of this algorithm in theorem 10.1.2.

Theorem 10.1.2. *Let $M \in \mathbb{Z}^{n\times n}$ be a square matrix with non-zero determinant. Then* `CalculateRepresentativesCosets`(M) *returns a set of representatives of the cosets of* $\operatorname{im}(M)$ *in* $\mathbb{Z}^n$.

Proof. Let $P, Q \in \mathrm{GL}_n(\mathbb{Z})$ such that $PMQ = S$, with S the Smith normal form of M. Since $\det(M) \neq 0$, we have that S is a diagonal matrix with non-zero entries on its diagonal, and we may assume that these entries are positive. We

Algorithm 5 Calculating representatives of the cosets of $\operatorname{im}(M)$ in $\mathbb{Z}^n$

function CalculateRepresentativesCosets(M)
  if $\det(M) = 0$ **then**
    return `fail`
  else
    $P, Q, S \leftarrow$ matrices s.t. $PMQ = S$, the Smith normal form of M
    $C \leftarrow \prod_{i=1}^{n} \{0, 1, \dots, S_{i,i} - 1\}$
    return $\{P^{-1}x \mid x \in C\}$
  end if
end function

use this to determine when two cosets of $\operatorname{im}(M)$ are equivalent:

$$\begin{aligned} x + \operatorname{im}(M) = y + \operatorname{im}(M) &\iff \exists z \in \mathbb{Z}^n : x - y = Mz \\ &\iff \exists z \in \mathbb{Z}^n : Px - Py = PMQ(Q^{-1}z) \\ &\iff \exists z' \in \mathbb{Z}^n : Px - Py = Sz' \\ &\iff Px + \operatorname{im}(S) = Py + \operatorname{im}(S) \\ &\iff \forall i \in \{1, \dots, n\} : (Px)_i \equiv (Py)_i \bmod S_{i,i}. \end{aligned}$$

Thus, the Cartesian product C defined by

$$C := \prod_{i=1}^{n} \{0, 1, \dots, S_{i,i} - 1\}$$

contains $\det(S) = \det(M)$ elements of $\mathbb{Z}^n$ that each represent a different coset of $\operatorname{im}(S)$ in $\mathbb{Z}^n$. Thus, each element of the set $P^{-1}C := \{P^{-1}x \mid x \in C\}$ represents a different coset of $\operatorname{im}(M)$ in $\mathbb{Z}^n$. □

Finally, algorithm 6 allows us to verify whether two elements of a crystallographic group are Reidemeister equivalent with respect to a given automorphism. We prove the correctness of this algorithm in theorem 10.1.3. Thus, combining algorithms 4 to 6 we may construct algorithm 7, which takes as input a crystallographic group Γ and an automorphism φ, and outputs the Reidemeister number $R(\varphi)$. Note that this algorithm actually calculates a complete set of representatives of the Reidemeister classes $\mathfrak{R}(\varphi)$.

Theorem 10.1.3. *Let Γ be a crystallographic group, $\varphi \in \operatorname{Aut}(\Gamma)$ and $\gamma_1, \gamma_2 \in \Gamma$. Then* `AreReidemeisterEquivalent`$(\Gamma, \varphi, \gamma_1, \gamma_2)$ *returns* `true` *if and only if $\gamma_1 \sim_\varphi \gamma_2$.*

Algorithm 6 Verifying if two elements γ_1, γ_2 of a crystallographic group Γ are Reidemeister equivalent

function AREREIDEMEISTEREQUIVALENT(Γ, φ, γ_1, γ_2)
    $d, D \leftarrow$ vector d, matrix D such that $\xi_{(d,D)} = \varphi$
    for $i \in \{1,2\}$ **do**
        $x_i, A_i \leftarrow$ vector x_i, matrix A_i such that $(x_i, A_i) = \gamma_i$
    end for
    for $(b,B) \in F_{ext}(\Gamma)$ **do**
        if $A_1 = BA_2DB^{-1}D^{-1}$ **then**
            if $(\mathbb{1}_n - A_1D)^{-1}(x_1 - Bx_2 - (BA_2 - A_1)d) - b \in \mathbb{Z}^n$ **then**
                return `true`
            end if
        end if
    end for
    return `false`
end function

Proof. Define x_i and A_i by $\gamma_i = (x_i, A_i) \in \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ for $i \in \{1,2\}$. We have that

$$\begin{aligned}\gamma_1 \sim_\varphi \gamma_2 &\iff \exists \delta \in \Gamma : \gamma_1 = \delta\gamma_2\varphi(\delta)^{-1}\\ &\iff \exists (y,B) \in \Gamma : (x_1, A_1) = (y,B)(x_2,A_2)(d,D)(y,B)^{-1}(d,D)^{-1}.\end{aligned}$$

Now let $b \in \mathbb{R}^n$ such that $(b,B) \in F_{ext}(\Gamma)$ and set $z = y - b \in \mathbb{Z}^n$. Splitting up in its components, we may rephrase the above condition as that there exist $(b,B) \in F_{ext}(\Gamma)$ and $z \in \mathbb{Z}^n$ such that

(a) $A_1 = BA_2DB^{-1}D^{-1}$,

(b) $x_1 = Bx_2 + (BA_2 - BA_2DB^{-1}D^{-1})d + (\mathbb{1}_n - BA_2DB^{-1})(z+b)$.

Suppose that condition (a) is true for some (b,B). Then we have left to verify that

$$\begin{aligned}&\exists z \in \mathbb{Z}^n : x_1 = Bx_2 + (BA_2 - BA_2DB^{-1}D^{-1})d + (\mathbb{1}_n - BA_2DB^{-1})(z+b)\\ \iff& \exists z \in \mathbb{Z}^n : x_1 = Bx_2 + (BA_2 - A_1)d + (\mathbb{1}_n - A_1D)(z+b)\\ \iff& \exists z \in \mathbb{Z}^n : x_1 - Bx_2 - (BA_2 - A_1)d = (\mathbb{1}_n - A_1D)(z+b)\\ \iff& \exists z \in \mathbb{Z}^n : (\mathbb{1}_n - A_1D)^{-1}(x_1 - Bx_2 - (BA_2 - A_1)d) = z + b\\ \iff& (\mathbb{1}_n - A_1D)^{-1}(x_1 - Bx_2 - (BA_2 - A_1)d) - b \in \mathbb{Z}^n,\end{aligned}$$

and this last line is exactly what the algorithm verifies. □

Algorithm 7 Calculating $R(\varphi)$ for an automorphism of a crystallographic group Γ

```
function ReidemeisterNumber(Γ, φ)
    ℜ(φ')_reps ← CalculateRepresentativesFiniteGroup(F, φ')
    ℜ(φ)_reps ← ∅
    D ← matrix D such that ξ_(d,D) = φ
    for A ∈ ℜ(φ')_reps do
        ℜ(AD)_reps ← CalculateRepresentativesCosets(𝟙_n − AD)
        a ← vector such that (a, A) ∈ F_ext(Γ)
        for x ∈ ℜ(AD)_reps do
            new ← true
            γ1 ← (x + a, A)
            for γ2 ∈ ℜ(φ)_reps do
                if AreReidemeisterEquivalent(Γ, φ, γ1, γ2) = true then
                    new ← false
                end if
            end for
            if new = true then
                ℜ(φ)_reps ← ℜ(φ)_reps ∪ {γ1}
            end if
        end for
    end for
    return #ℜ(φ)_reps
end function
```

Constructing a set of representatives of $\mathrm{Out}(\Gamma)$

While we can now calculate the Reidemeister number for a given automorphism $\varphi = \xi_{(d,D)}$ with algorithm 7, we still require a set of representatives of $\mathrm{Out}(\Gamma)$ to apply it to. The following theorem helps understand the structure of the outer automorphism group.

Theorem 10.1.4 (see [Cha86, Theorem V.1.1]). *Let Γ be a crystallographic group with holonomy group $F \subseteq \mathrm{GL}_n(\mathbb{Z})$. Then the following diagram commutes and all columns and rows are short exact sequences:*

$$
\begin{array}{ccccccccccc}
 & & 1 & & 1 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
1 & \longrightarrow & Z(\Gamma) & \longrightarrow & Z(\Gamma) & \longrightarrow & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z(\Gamma) & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathrm{Inn}(\Gamma) & \longrightarrow & F & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & \longrightarrow & \mathrm{Aut}^0(\Gamma) & \longrightarrow & \mathrm{Aut}(\Gamma) & \longrightarrow & N_\Gamma & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & \longrightarrow & H^1(F,\mathbb{Z}^n) & \xrightarrow{i} & \mathrm{Out}(\Gamma) & \xrightarrow{p} & N_\Gamma/F & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & & 1 & &
\end{array}
$$

We define $\mathrm{Aut}^0(\Gamma)$ *in the diagram above as the (abelian) group of automorphisms* φ *of* Γ *that satisfy* $\varphi|_{\mathbb{Z}^n} = \mathbb{1}_n$.

We are, of course, particularly interested in the bottom row of this diagram. Algorithm 1 lets us calculate N_Γ (and by extension N_Γ/F). Moreover, since this algorithm returns a d such that $\xi_{(d,D)} \in \mathrm{Aut}(\Gamma)$ for a given $D \in N_\Gamma$, we can actually construct a (representative of a) preimage under the projection p for any element of N_Γ/F. Thus, we are left to calculate the first cohomology group $H^1(F, \mathbb{Z}^n)$. This can be done using algorithm 8, which was also described in [Lut13, Section 4.2]. We prove the correctness of this algorithm in theorem 10.1.5.

Algorithm 8 Calculating representatives of $H^1(F, \mathbb{Z}^n)$

---

function CalculateRepresentativesCohomology(F)

$M \leftarrow \begin{pmatrix} \mathbb{1}_n - A_1 \\ \mathbb{1}_n - A_2 \\ \vdots \\ \mathbb{1}_n - A_k \end{pmatrix}$ $\qquad \triangleright\ F = \{A_1, \dots, A_k\}$

$P, Q, S \leftarrow$ matrices such that $PMQ = S$, the Smith normal form of M

$r \leftarrow \mathrm{rank}(S)$

$C \leftarrow \prod_{i=1}^{r} \left\{0, \frac{1}{S_{i,i}}, \frac{2}{S_{i,i}}, \dots, \frac{S_{i,i}-1}{S_{i,i}}\right\} \times \prod_{i=r+1}^{n} \{0\}$

return $\{Qd' \mid d' \in C\}$

end function

Theorem 10.1.5. *Let $F \subseteq \mathrm{GL}_n(\mathbb{Z})$ be the holonomy group of a crystallographic group. Then* `CalculateRepresentativesCohomology`(F) *returns a set of representatives of $H^1(F, \mathbb{Z}^n)$.*

Proof. Let $d \in \mathbb{R}^n$. We start by finding necessary and sufficient conditions on d such that $\xi_{(d,\mathbb{1}_n)}$ is an automorphism, i.e. $\xi_{(d,\mathbb{1}_n)} \in \mathrm{Aut}^0(\Gamma)$. It must satisfy $(d, \mathbb{1}_n)(a_i, A_i)(d, \mathbb{1}_n)^{-1} \in \Gamma$ for every $i \in \{1, \ldots, k\}$, or equivalently

$$\begin{aligned} &(d, \mathbb{1}_n)(a_i, A_i)(d, \mathbb{1}_n)^{-1} \in \Gamma \\ \iff & (d, \mathbb{1}_n)(a_i, A_i)(d, \mathbb{1}_n)^{-1}(a_i, A_i)^{-1} \in \mathbb{Z}^n \\ \iff & (\mathbb{1}_n - A_i)d \in \mathbb{Z}^n. \end{aligned} \tag{10.1}$$

Therefore, construct the matrix

$$M := \begin{pmatrix} \mathbb{1}_n - A_1 \\ \mathbb{1}_n - A_2 \\ \vdots \\ \mathbb{1}_n - A_k \end{pmatrix} \in \mathbb{Z}^{nk \times n},$$

and calculate the matrices $P \in \mathrm{GL}_{nk}(\mathbb{Z})$, $S \in \mathbb{Z}^{nk \times n}$ and $Q \in \mathrm{GL}_n(\mathbb{Z})$ such that S is the Smith normal form of M and $PMQ = S$. Define $d' := Q^{-1}d$ and observe that condition (10.1) holding for every $i \in \{1, \ldots, k\}$ is equivalent to

$$Sd' \in \mathbb{Z}^{nk}. \tag{10.2}$$

If we set $r = \mathrm{rank}(S)$, then for the coordinates d'_i of d' this means that $d'_i \in \frac{1}{S_{i,i}}\mathbb{Z}$ for $i \in \{1, \ldots, r\}$. The other coordinates of d' have no restrictions imposed on them.

Now, suppose we have a $d \in \mathbb{R}^n$ such that $d' = Q^{-1}d$ satisfies criterion (10.2). We decompose d in three vectors:

$$d = Q \begin{pmatrix} d'_1 \\ \vdots \\ d'_r \\ d'_{r+1} \\ \vdots \\ d'_n \end{pmatrix} = Q \begin{pmatrix} d'_1 - \lfloor d'_1 \rfloor \\ \vdots \\ d'_r - \lfloor d'_r \rfloor \\ 0 \\ \vdots \\ 0 \end{pmatrix} + Q \begin{pmatrix} \lfloor d'_1 \rfloor \\ \vdots \\ \lfloor d'_r \rfloor \\ 0 \\ \vdots \\ 0 \end{pmatrix} + Q \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d'_{r+1} \\ \vdots \\ d'_n \end{pmatrix}.$$

For the first vector, which we will call d^{base}, we must have that

$$d'_i - \lfloor d'_i \rfloor \in \left\{0, \frac{1}{S_{i,i}}, \frac{2}{S_{i,i}}, \ldots, \frac{S_{i,i} - 1}{S_{i,i}}\right\}.$$

The second vector, which we will call d^{int}, is an element of $\mathbb{Z}^n$, and the final vector, which we will call d^{rem}, satisfies

$$(d^{rem}, \mathbb{1}_n)(a_i, A_i)(d^{rem}, \mathbb{1}_n)^{-1} = (a_i, A_i)$$

for all $i \in \{1, \dots, k\}$. Thus, we may conclude that every $\xi_{(d,\mathbb{1}_n)} \in \mathrm{Aut}^0(\Gamma)$ admits a unique decomposition

$$\begin{aligned}\xi_{(d,\mathbb{1}_n)} &= \xi_{(d^{base},\mathbb{1}_n)} \circ \xi_{(d^{int},\mathbb{1}_n)} \circ \xi_{(d^{rem},\mathbb{1}_n)} \\ &= \xi_{(d^{base},\mathbb{1}_n)} \circ \iota \circ \mathrm{id},\end{aligned}$$

where $\iota = \xi_{(d^{int},\mathbb{1}_n)}$ lies in the image of the map $\mathbb{Z}^n \to \mathrm{Aut}^0(\Gamma)$ in the diagram in theorem 10.1.4. Thus, we find that

$$\left\{ Qd' \;\middle|\; \begin{array}{ll} d'_i \in \left\{0, \frac{1}{S_{i,i}}, \frac{2}{S_{i,i}}, \dots, \frac{S_{i,i}-1}{S_{i,i}}\right\} & \text{for } 1 \leq i \leq r \\ d'_i = 0 & \text{for } r+1 \leq i \leq n \end{array} \right\}$$

is a set of representatives of $H^1(F, \mathbb{Z}^n)$. □

Calculating the Reidemeister spectrum

Combining the algorithms from the previous sections, we may now construct algorithm 9, which calculates the Reidemeister spectrum of a crystallographic group Γ with finite outer automorphism group. A GAP-implementation of algorithm 9 produced the results found in tables B.1 to B.6. Note that we have omitted the value ∞ from the Reidemeister spectra in these tables. The Bieberbach groups are indicated by a star (*).

10.1.2 Infinite outer automorphism group

The Reidemeister spectra of crystallographic groups with infinite outer automorphism groups have to be calculated by hand. For dimensions 1, 2 and 3, we do this for all groups; for dimension 4 we limit ourselves to a small selection of groups, for example those where we can apply lemma 2.5.18 or theorem 4.2.6.

Dimension 2

min.2-1.1-0. This group is isomorphic to $\mathbb{Z}^2$, hence by theorem 5.1.2 we find that $\mathrm{Spec}_R(\Gamma) = \mathbb{N} \cup \{\infty\}$.

Algorithm 9 Calculate the Reidemeister spectrum of a crystallographic group Γ with finite outer automorphism group

```
function CalculateReidemeisterSpectrum(Γ)
    N_F ← N_{GL_n(ℤ)}(F)
    if #N_F = ∞ then
        return fail                                              ▷ # Out(Γ) = ∞
    else
        H_reps ← CalculateRepresentativesCohomology(F)
        Out_reps ← ∅                                             ▷ Calculate Out(Γ)
        for [D] ∈ N_F/F do
            d^base ← ExtendsToAutomorphism(D, Γ)
            if d^base ≠ fail then
                Out_reps ← Out_reps ∪ {ξ_(d^base + d, D) | d ∈ H_reps}
            end if
        end for
        Spec_R(Γ) ← ∅                                            ▷ Calculate Spec_R(Γ)
        for φ ∈ Out_reps do
            Spec_R(Γ) ← Spec_R(Γ) ∪ ReidemeisterNumber(Γ, φ)
        end for
        return Spec_R(Γ)
    end if
end function
```

group.1-1.1-0. This group is isomorphic to $\langle \mathbb{Z}^2, (0, -\mathbb{1}_2)\rangle$, hence by theorem 7.1.3 we find that $\mathrm{Spec}_R(\Gamma) = 2\mathbb{N} \cup \{3, \infty\}$.

Dimension 3

min.6-1.1-0. This group is isomorphic to $\mathbb{Z}^3$, hence by theorem 5.1.2 we find that $\mathrm{Spec}_R(\Gamma) = \mathbb{N} \cup \{\infty\}$.

min.7-1.1-0. This group is isomorphic to the direct product of min.1-1.1-0 and group.1-1.1-0, where both factors are characteristic. Hence by lemma 2.5.18 we find that $\mathrm{Spec}_R(\Gamma) = 4\mathbb{N} \cup \{6, \infty\}$. This group is also isomorphic to $\Lambda_{3/2/0}$, whose spectrum was calculated in table 7.1.

min.7-1.1-1. This group is isomorphic to $\Lambda_{3/2/1}$, hence by theorem 7.1.9 we find that $\mathrm{Spec}_R(\Gamma) = 2\mathbb{N} \cup \{\infty\}$.

min.7-1.2-0. This group is given by

$$\Gamma = \langle \mathbb{Z}^3, \alpha \rangle \text{ with } \alpha = (0, \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}).$$

We first calculate an explicit formula for the Reidemeister number of a given automorphism.

Proposition 10.1.6. *Let Γ be the crystallographic group min.7-1.2-0 and $\varphi = \xi_{(d,D)} \in \operatorname{Aut}(\Gamma)$. Then*

$$R(\varphi) = \left(\frac{1}{\#F} \sum_{A \in F} |\det(\mathbb{1}_3 - AD)|_\infty \right) + 4\delta(d),$$

with δ given by

$$\delta(d) := \begin{cases} 1 & \text{if } d_3 \in \mathbb{Z}, \\ 0 & \text{otherwise} \end{cases},$$

where d_3 is the third coordinate of d.

Proof. Let $\varphi = \xi_{(d,D)}$ be an automorphism of Γ. We can calculate that $Z(\Gamma) = \langle e_1 \rangle$ and define $\Gamma' := \Gamma/Z(\Gamma)$, which is the crystallographic group group.1-1.1-0. Then φ induces an automorphism $\varphi' = \xi_{(d',D')}$ on Γ'. One can verify, using that D commutes with any element of the holonomy group, that we may assume D and d are of the form

$$D = \begin{pmatrix} \varepsilon & m_1 \quad m_3 \\ 0 & D' \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ d' \end{pmatrix},$$

where

$$D' = \begin{pmatrix} \varepsilon + 2m_1 & 2m_3 \\ m_2 & 1 + 2m_4 \end{pmatrix}, \quad d' = \begin{pmatrix} d_2 \\ d_3 \end{pmatrix}.$$

Here, $\varepsilon \in \{-1, 1\}$, $m_1, m_2, m_3, m_4, d_2 \in \mathbb{Z}$, and importantly, $d_3 \in \frac{1}{2}\mathbb{Z}$. For $A \in F$, let A' be the projection to the holonomy group F' of Γ'. We have that

$$|\det(\mathbb{1}_3 - AD)|_\infty = |1 - \varepsilon|_\infty |\det(\mathbb{1}_2 - A'D')|_\infty. \tag{10.3}$$

Following theorem 4.2.5 and the first part of the proof of theorem 7.1.3, we may conclude that $R(\varphi) = \infty$ if and only if (at least) one of the following three conditions is satisfied:

- $\varepsilon = 1$,

- $\det(D') = -1$ and $\operatorname{tr}(D') = 0$,
- $\det(D') = 1$ and $|\operatorname{tr}(D')| = 2$.

If this is the case, then the formula holds. We are left to verify the formula when none of these conditions are satisfied.

Consider a Reidemeister class $[x]_\varphi$ of Γ and recall that $Z(\Gamma) = \langle e_1 \rangle$. Then

$$x = e_1^{-k}(xe_1^{2k})\varphi(e_1^{-k})^{-1}$$

and hence $x \sim xe_1^{2k}$ for all $k \in \mathbb{Z}$. So a Reidemeister class $[xZ(\Gamma)]_{\varphi'}$ of Γ' lifts to at most 2 distinct Reidemeister classes of Γ: $[x]_\varphi$ and $[xe_1]_\varphi$.

The question that remains is: when is $x \sim_\varphi xe_1$? This is the case when there exists some $z \in \Gamma$ such that

$$x = zxe_1\varphi(z)^{-1}. \tag{10.4}$$

Projecting this to Γ' we find

$$xZ(\Gamma) = zx\varphi(z)^{-1}Z(\Gamma). \tag{10.5}$$

Set $x = ((x_1, x_2, x_3)^\intercal, A_x)$ and $z = ((z_1, z_2, z_3)^\intercal, A_z)$. If we assume that $A_z = \mathbb{1}_3$, then (10.5) is equivalent to

$$(\mathbb{1}_2 - A'_x D')\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = 0.$$

But $\det(\mathbb{1}_2 - A'_x D') \neq 0$, hence $z_2 = z_3 = 0$ and thus $z = e_1^{z_1} \in Z(\Gamma)$ for some $z_1 \in \mathbb{Z}$. But then equation (10.4) reduces to $e_1^{2z_1+1} = 1$, which is impossible. Therefore, $A_z \neq \mathbb{1}_3$, and then equation (10.5) is a special case of equation (7.2): $[xZ(\Gamma)]_{\varphi'}$ is one of the cosets of $\mathbb{1}_2 - A'_x D'$ such that

$$2\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + 2A'_x\begin{pmatrix} d_2 \\ d_3 \end{pmatrix} = (\mathbb{1}_2 - A'_x D')\begin{pmatrix} z_2 \\ z_3 \end{pmatrix}, \tag{10.6}$$

i.e. a coset that forms a Reidemeister class on its own, rather than pairing up with another coset. The e_2- and e_3-coordinates of (10.4) are equivalent to equation (10.6). The e_1-coordinate can be shown to be equivalent to $z_2 = 2z_1+1$, under the assumption that (10.6) is satisfied. But since z_1 is an integer, we need that $z_2 \in 2\mathbb{Z} + 1$.

Now, let's look at the number of Reidemeister classes $[xZ(\Gamma)]_{\varphi'}$ such that (10.6) holds. From the calculations we did for Γ', we know that we must look at the number of solutions $O(\mathbb{1}_2 - D', 2d')$ of the system of equations over $\mathbb{Z}_2$ given by

$$\begin{pmatrix} 0 & 0 \\ \bar{m}_2 & 0 \end{pmatrix}\begin{pmatrix} \bar{z}_2 \\ \bar{z}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{2d_3} \end{pmatrix},$$

and we see that $O(\mathbb{1}_2 - D', 2d') = 2\,O(m_2, 2d_3)$, since $\bar{z}_3$ can be chosen freely. We now have 4 cases:

1. $\bar{m}_2 = \bar{0}, \overline{2d_3} = \bar{0}$. Then $O(m_2, 2d_3) = 2$ with solutions $\bar{z}_2 = \bar{0},\ \bar{1}$; and $\delta(d) = 1$.
2. $\bar{m}_2 = \bar{1}, \overline{2d_3} = \bar{0}$. Then $O(m_2, 2d_3) = 1$ with solution $\bar{z}_2 = \bar{0}$; and $\delta(d) = 1$.
3. $\bar{m}_2 = \bar{0}, \overline{2d_3} = \bar{1}$. Then $O(m_2, 2d_3) = 0$; and $\delta(d) = 0$.
4. $\bar{m}_2 = \bar{1}, \overline{2d_3} = \bar{1}$. Then $O(m_2, 2d_3) = 1$ with solution $\bar{z}_2 = \bar{1}$; and $\delta(d) = 0$.

Every solution $\bar{z}_2$ of the equation $\bar{m}_2\bar{z}_2 = \overline{2d_3}$ represents 4 Reidemeister classes $[xZ(\Gamma)]_{\varphi'}$, since one takes all combinations of $\bar{z}_3 \in \{0, 1\}$ and $A'_x \in \{\mathbb{1}_2, -\mathbb{1}_2\}$. Thus, we have respectively 8, 4, 0 and 4 Reidemeister classes $[xZ(\Gamma)]_{\varphi'}$ satisfying (10.5); of which respectively 4, 0, 0 and 4 satisfy $z_2 \in 2\mathbb{Z} + 1$. So the number of lifts to Reidemeister classes of Γ is respectively 12, 8, 0 and 4. This number of Reidemeister classes always equals

$$2\,O(\mathbb{1}_2 - D', 2d') + 4\delta(d).$$

On the other hand, Γ' has

$$\frac{|\det(\mathbb{1}_2 - D')| + |\det(\mathbb{1}_2 + D')|}{2} - O(\mathbb{1}_2 - D', 2d')$$

Reidemeister classes for which (10.5) does not hold, meaning each of these classes lift to two distinct Reidemeister classes of Γ. Combining all the classes we obtain the formula

$$R(\varphi) = |\det(\mathbb{1}_2 - D')| + |\det(\mathbb{1}_2 + D')| + 4\delta(d), \tag{10.7}$$

and using $1 - \varepsilon = 2$ in equation (10.3) we see this is exactly

$$\begin{aligned} R(\varphi) &= \left(\frac{1}{\#F} \sum_{A \in F} |\det(\mathbb{1}_3 - AD)| \right) + 4\delta(d) \\ &= \left(\frac{1}{\#F} \sum_{A \in F} |\det(\mathbb{1}_3 - AD)|_\infty \right) + 4\delta(d), \end{aligned}$$

since none of the determinants are zero. Therefore, the proposed formula holds in all cases. □

Theorem 10.1.7. *Let Γ be the crystallographic group min.7-1.2-0. Then* $\mathrm{Spec}_R(\Gamma) = 4\mathbb{N} \cup \{\infty\}$.

Proof. Let $\varphi = \xi_{(d,D)}$ be an automorphism of Γ with $R(\varphi) < \infty$. Consider formula (10.7) and remark that $\operatorname{tr}(D') \in 2\mathbb{Z}$. Since

$$\det(\mathbb{1}_2 \pm D') = 1 \pm \operatorname{tr}(D') + \det(D'),$$

we have that

$$|\det(\mathbb{1}_2 - D')| + |\det(\mathbb{1}_2 + D')| = \begin{cases} 4 & \text{if } \operatorname{tr}(D') = 0, \det(D') = 1, \\ 2|\operatorname{tr}(D')| & \text{otherwise ,} \end{cases}$$

so in both cases $R(\varphi) \in 4\mathbb{N}$. Now consider the family of automorphisms $\varphi_m = \xi_{(d,D_m)}$ given by

$$D_m = \begin{pmatrix} -1 & m & m \\ 0 & -1+2m & 2m \\ 0 & 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix},$$

where $m \in \mathbb{N}$. Since $\det(\mathbb{1}_2 \pm D') = \pm 2m$ and $\delta(d) = 0$, we find that $R(\varphi_m) = 4m$ and hence $\operatorname{Spec}_R(\Gamma) = 4\mathbb{N} \cup \{\infty\}$. □

group.5-1.1-0. This group is isomorphic to $\langle \mathbb{Z}^3, (0, -\mathbb{1}_3)\rangle$, hence by theorem 7.1.3 we find that $\operatorname{Spec}_R(\Gamma) = \mathbb{N} \setminus \{1\} \cup \{\infty\}$.

Dimension 4

min.15-1.1-0. This group is isomorphic to $\mathbb{Z}^4$, hence by theorem 5.1.2 we find that $\operatorname{Spec}_R(\Gamma) = \mathbb{N} \cup \{\infty\}$.

min.17-1.1-0. This group is isomorphic to the direct product of min.1-1.1-0 and group.5-1.1-0, where both factors are characteristic. Hence by lemma 2.5.18 we find that $\operatorname{Spec}_R(\Gamma) = 2\mathbb{N} \setminus \{2\} \cup \{\infty\}$. This group is also isomorphic to $\Lambda_{4/3/0}$, whose spectrum was calculated in table 7.1.

min.17-1.1-1. This group is isomorphic to $\Lambda_{4/3/1}$, hence by theorem 7.1.9 we find that $\operatorname{Spec}_R(\Gamma) = 2\mathbb{N} \cup \{\infty\}$.

min.18-1.1-0. This group is isomorphic to the direct product of min.2-1.1-0 and group.5-1.1-0, where both factors are characteristic. Hence by lemma 2.5.18 we find that $\operatorname{Spec}_R(\Gamma) = 2\mathbb{N} \cup 3\mathbb{N} \cup \{\infty\}$. This group is also isomorphic to $\Lambda_{4/2/0}$, whose spectrum was calculated in table 7.1.

min.18-1.1-1. This group is isomorphic to $\Lambda_{4/2/1}$, hence by theorem 7.1.9 we find that $\mathrm{Spec}_R(\Gamma) = 2\mathbb{N} \cup \{\infty\}$.

min.18-1.2-1. This is a Bieberbach group given by

$$\Gamma = \langle \mathbb{Z}^4, \alpha \rangle \text{ with } \alpha = (\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}).$$

Theorem 10.1.8. *Let Γ be the crystallographic group min.18-1.2-1. Then $\mathrm{Spec}_R(\Gamma) = 4\mathbb{N} \cup \{\infty\}$.*

Proof. Let $\varphi = \xi_{(d,D)}$ be an automorphism of Γ. Note that $Z(\Gamma) = \langle e_1, e_2 \rangle$ and $\mathbb{Z}^n \cap \sqrt[\Gamma]{\gamma_2(\Gamma)} = \langle e_2 e_3^2, e_4 \rangle$, and these are both characteristic subgroups of Γ. Taking into account that $\det(D) \in \{-1, 1\}$, one can then calculate that D must be of the form

$$D = \begin{pmatrix} D_1 & D_2 \\ 0 & D_3 \end{pmatrix} = \begin{pmatrix} 2m_1 + 1 & 2m_3 & -m_3 & 0 \\ m_2 & 2m_4 + 1 & -m_4 + m_5 & m_7 \\ 0 & 0 & 2m_5 + 1 & 2m_7 \\ 0 & 0 & m_6 & 2m_8 + 1 \end{pmatrix},$$

with all $m_i \in \mathbb{Z}$. Using the averaging formula from theorem 4.2.6, we find that

$$R(\varphi) = \frac{1}{2} |\det(\mathbb{1}_2 - D_1)|_\infty \left(|\det(\mathbb{1}_2 - D_3)|_\infty + |\det(\mathbb{1}_2 + D_3)|_\infty\right).$$

Now, assuming that $R(\varphi) < \infty$, we have

$$\det(\mathbb{1}_2 - D_1) = \det \begin{pmatrix} -2m_1 & -2m_3 \\ -m_2 & -2(m_4 + m_5) \end{pmatrix} \in 2\mathbb{N},$$

and, noting that $\mathrm{tr}(D_3) \in 2\mathbb{N}$, we find

$$|\det(\mathbb{1}_2 - D_3)| + |\det(\mathbb{1}_2 + D_3)| = \begin{cases} 4 & \text{if } \mathrm{tr}(D_3) = 0, \det(D_3) = 1, \\ 2|\mathrm{tr}(D_3)| & \text{otherwise,} \end{cases}$$

hence $|\det(\mathbb{1}_2 - D_3)| + |\det(\mathbb{1}_2 + D_3)| \in 4\mathbb{N}$. Thus, putting everything together we find that $R(\varphi) \in 4\mathbb{N}$. Now consider the family of automorphisms φ_m given by

$$\varphi_m(e_1) = e_1^{-1} e_2, \qquad \varphi_m(e_4) = e_2 e_3^2 e_4^{-1},$$

$$\varphi_m(e_2) = e_1^{2m} e_2^{-2m+1}, \qquad \varphi_m(\alpha) = e_3 \alpha^{-1},$$

$$\varphi_m(e_3) = e_1^{-m} e_2^{m-1} e_3^{-1} e_4,$$

which have associated matrix

$$D_m = \begin{pmatrix} -1 & 2m & -m & 0 \\ 1 & -2m+1 & m-1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

for every $m \in \mathbb{N}$. Then $R(\varphi_m) = 4m$ and hence $\mathrm{Spec}_R(\Gamma) = 4\mathbb{N} \cup \{\infty\}$. □

group.26-1.1-0. This group is isomorphic to $\langle \mathbb{Z}^4, (0, -\mathbb{1}_4)\rangle$, hence by theorem 7.1.3 we find that $\mathrm{Spec}_R(\Gamma) = \mathbb{N} \setminus \{1\} \cup \{\infty\}$.

group.179-1.1-0. This group is isomorphic to the direct product of min.2-1.1-0 and min.5-1.1-0, where both factors are characteristic. Hence by lemma 2.5.18 we find that $\mathrm{Spec}_R(\Gamma) = 4\mathbb{N} \cup \{\infty\}$.

group.179-1.1-1. This is a Bieberbach group given by

$$\Gamma = \langle \mathbb{Z}^4, \alpha\rangle \text{ with } \alpha = (\begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}).$$

Theorem 10.1.9. *Let Γ be the crystallographic group group.179-1.1-1. Then $\mathrm{Spec}_R(\Gamma) = 6\mathbb{N} \cup \{\infty\}$.*

Proof. Let $\varphi = \xi_{(d,D)}$ be an automorphism of Γ. Note that $Z(\Gamma) = \langle e_1, e_2\rangle$ and $\mathbb{Z}^n \cap \sqrt[\Gamma]{\gamma_2(\Gamma)} = \langle e_3, e_4\rangle$, and these are both characteristic subgroups of Γ. Moreover, $\Gamma' := \Gamma/Z(\Gamma)$ is the crystallographic group min.5-1.1-0. Let $\varphi' = \xi_{(d',D')}$ be the induced automorphism on Γ'. Because $N_{\Gamma'}$ is finite, we can calculate (for example with a computer) that

- $\displaystyle\sum_{A' \in F'} |\det(\mathbb{1}_2 - A'D')|_\infty \in \{6, \infty\}$ for all $D' \in N_{\Gamma'}$,
- if the above sum is finite, then $D'A'D'^{-1} = A'$ for all $A' \in F'$.

If we assume that $R(\varphi) < \infty$, we must therefore have that $DAD^{-1} = A$ for all $A \in F$. One can then calculate that D must be of the form

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D' \end{pmatrix} = \begin{pmatrix} 3m_1+1 & m_3 & 0 & 0 \\ 3m_2 & m_4 & 0 & 0 \\ 0 & 0 & m_5 & m_7 \\ 0 & 0 & m_6 & m_8 \end{pmatrix},$$

with all $m_i \in \mathbb{Z}$ and $D' \in N_{\Gamma'}$. Using the averaging formula from theorem 4.2.6, we find that

$$R(\varphi) = \frac{1}{3} |\det(\mathbb{1}_2 - D_1)|_\infty \sum_{A' \in F'} |\det(\mathbb{1}_2 - A'D')|_\infty.$$

The first column of D_1 tells us that $\det(\mathbb{1}_2 - D_1) \in 3\mathbb{N}$, and we already established that the sum over F' must equal 6. Thus, we find that $R(\varphi) \in 6\mathbb{N}$. Now consider the family of automorphisms φ_m given by

$$\varphi_m(e_1) = e_1 e_2^{3m}, \qquad \varphi_m(e_4) = e_4^{-1},$$

$$\varphi_m(e_2) = e_1 e_2^{3m-1}, \qquad \varphi_m(\alpha) = e_2^m e_3 \alpha,$$

$$\varphi_m(e_3) = e_3^{-1},$$

which have associated matrix

$$D_m = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 3m & 3m-1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

for every $m \in \mathbb{N}$. Then $R(\varphi_m) = 6m$ and hence $\mathrm{Spec}_R(\Gamma) = 6\mathbb{N} \cup \{\infty\}$. □

group.179-1.2-1. This is a Bieberbach group given by

$$\Gamma = \langle \mathbb{Z}^4, \alpha \rangle \text{ with } \alpha = (\begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}).$$

Theorem 10.1.10. *Let Γ be the crystallographic group group.179-1.2-1. Then* $\mathrm{Spec}_R(\Gamma) = 6\mathbb{N} \cup \{\infty\}$.

Proof. Let $\varphi = \xi_{(d,D)}$ be an automorphism of Γ. Assume that $R(\varphi) < \infty$, then by following the same reasoning as for group.179-1.1-1, one can calculate that D must be of the form

$$D = \begin{pmatrix} D_1 & D_2 \\ 0 & D' \end{pmatrix} = \begin{pmatrix} 3m_1 + 1 & 3m_3 & m_5 & m_9 \\ m_2 & m_4 & m_6 & m_{10} \\ 0 & 0 & m_7 & m_{11} \\ 0 & 0 & m_8 & m_{12} \end{pmatrix},$$

with all $m_i \in \mathbb{Z}$ and $D' \in N_{\Gamma'}$, where $\Gamma' := \Gamma/Z(\Gamma)$ is the crystallographic group min.5-1.1-0. Again, we find that $R(\varphi) \in 6\mathbb{N}$. Now consider the family of automorphisms φ_m given by

$$\varphi_m(e_1) = e_1e_2, \qquad \varphi_m(e_4) = e_4^{-1},$$

$$\varphi_m(e_2) = e_1^{3m}e_2^{3m-1}, \qquad \varphi_m(\alpha) = e_3\alpha,$$

$$\varphi_m(e_3) = e_3^{-1},$$

which have associated matrix

$$D_m = \begin{pmatrix} 1 & 3m & 0 & 0 \\ 1 & 3m-1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

for every $m \in \mathbb{N}$. Then $R(\varphi_m) = 6m$ and hence $\mathrm{Spec}_R(\Gamma) = 6\mathbb{N} \cup \{\infty\}$. □

group.182-1.1-0. This group is isomorphic to the direct product of group.5-1.1-0 and min.5-1.1-0, where both factors are characteristic. Hence by lemma 2.5.18 we find that $\mathrm{Spec}_R(\Gamma) = 8\mathbb{N} \cup \{12, \infty\}$.

10.1.3 Summary

Below, we present a table containing all crystallographic groups of dimension at most 4 that do not have the R_∞-property. The table also contains the sizes of the outer automorphism groups and the Reidemeister spectra. Note that we have omitted the value $\{\infty\}$ from the spectra, and have indicated Bieberbach groups with a star (*).

CARAT	BBNWZ	IT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
min.1-1.1-0*	1/1/1/1/1	1/1	2	$\{2\}$
min.2-1.1-0*	2/1/1/1/1	2/1	∞	$\mathbb{N}$
min.5-1.1-0	2/4/1/1/1	2/13	12	$\{4\}$
min.6-1.1-0*	3/1/1/1/1	3/1	∞	$\mathbb{N}$
min.7-1.1-0	3/2/1/1/1	3/3	∞	$4\mathbb{N} \cup \{6\}$
min.7-1.1-1*	3/2/1/1/2	3/4	∞	$2\mathbb{N}$
min.7-1.2-0	3/2/1/2/1	3/5	∞	$4\mathbb{N}$
min.10-1.1-0	3/3/1/1/1	3/16	96	$\{2\}$
min.10-1.1-3*	3/3/1/1/2	3/19	96	$\{2\}$

Table 10.1: Crystallographic groups of dimension at most 4 that do not have the R_∞-property (we omit ∞ from the spectra)

CARAT	BBNWZ	IT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
min.10-1.3-0	3/3/1/3/1	3/22	48	$\{2\}$
min.10-1.4-0	3/3/1/4/1	3/23	48	$\{2\}$
min.10-1.4-1	3/3/1/4/2	3/24	48	$\{2\}$
min.13-1.1-0	3/5/1/2/1	3/143	24	$\{8\}$
min.13-1.2-0	3/5/1/1/1	3/146	4	$\{8\}$
min.15-1.1-0*	4/1/1/1/1		∞	$\mathbb{N}$
min.17-1.1-0	4/2/2/1/1		∞	$2\mathbb{N} \setminus \{2\}$
min.17-1.1-1*	4/2/2/1/2		∞	$2\mathbb{N}$
min.17-1.2-0	4/2/2/2/1		∞	
min.18-1.1-0	4/3/1/1/1		∞	$2\mathbb{N} \cup 3\mathbb{N}$
min.18-1.1-1*	4/3/1/1/2		∞	$2\mathbb{N}$
min.18-1.2-0	4/3/1/2/1		∞	
min.18-1.2-1*	4/3/1/2/2		∞	$4\mathbb{N}$
min.18-1.3-0	4/3/1/3/1		∞	
min.28-1.1-0	4/22/7/2/1		8	$\{12\}$
min.32-1.1-0	4/22/1/2/1		288	$\{4, 16\}$
min.32-1.2-0	4/22/1/1/1		48	$\{16\}$
min.36-1.1-0	4/10/1/1/1		∞	
min.38-1.1-0	4/32/10/2/1		144	$\{6\}$
min.38-1.1-4	4/32/10/2/7		24	$\{6\}$
min.43-1.1-0	4/28/1/1/1		∞	
min.44-1.1-0	4/28/2/1/1		∞	
max.6-1.1-0	4/26/2/1/1		∞	
max.6-1.1-1	4/26/2/1/2		∞	
group.1-1.1-0	2/1/2/1/1	2/2	∞	$2\mathbb{N} \cup \{3\}$
group.5-1.1-0	3/1/2/1/1	3/2	∞	$\mathbb{N} \setminus \{1\}$
group.26-1.1-0	4/1/2/1/1		∞	$\mathbb{N} \setminus \{1\}$
group.28-1.1-0	4/3/2/1/1		∞	
group.28-1.1-1	4/3/2/1/2		∞	
group.28-1.1-2	4/3/2/1/3		∞	
group.28-1.2-0	4/3/2/2/1		∞	
group.28-1.2-1	4/3/2/2/2		∞	
group.28-1.2-2	4/3/2/2/3		∞	
group.28-1.3-0	4/3/2/3/1		∞	
group.37-1.1-0	4/21/2/2/1		12	$\{3\}$
group.40-1.1-0	4/22/2/2/1		16	$\{8\}$
group.44-1.1-0	4/22/5/4/1		16	$\{6\}$
group.44-3.1-0	4/22/5/3/1		144	$\{6\}$
group.52-1.1-0	4/5/1/2/1		192	$\{4\}$
group.52-1.1-6*	4/5/1/2/9		192	$\{4\}$

Table 10.1: Crystallographic groups of dimension at most 4 that do not have the R_∞-property (we omit ∞ from the spectra)

CARAT	BBNWZ	IT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.52-1.3-0	4/5/1/9/1		96	{4}
group.52-1.6-0	4/5/1/13/1		48	{4}
group.52-1.7-0	4/5/1/5/1		96	{4}
group.52-1.7-1	4/5/1/5/2		96	{4}
group.52-1.12-0	4/5/1/7/1		96	{4}
group.52-1.12-3*	4/5/1/7/4		96	{4}
group.52-1.13-0	4/5/1/1/1		12	{4}
group.96-1.1-0	4/16/1/1/1		∞	
group.96-1.1-1	4/16/1/1/2		∞	
group.96-2.1-0	4/16/1/2/1		∞	
group.96-2.1-1	4/16/1/2/2		∞	
group.96-2.1-2	4/16/1/2/3		∞	
group.96-3.1-0	4/16/1/3/1		∞	
group.78-1.1-0	4/32/4/2/1		48	{2, 6}
group.78-1.1-2	4/32/4/2/3		48	{2, 6}
group.78-1.1-4	4/32/4/2/6		24	{2, 6}
group.80-1.1-0	4/5/2/2/1		768	{2, 4}
group.80-1.1-5	4/5/2/2/16		256	{2, 4}
group.80-1.1-18	4/5/2/2/18		128	{2, 4}
group.80-1.4-0	4/5/2/9/1		192	{4}
group.80-1.4-2	4/5/2/9/3		64	{4}
group.80-1.6-0	4/5/2/6/1		64	{4}
group.80-1.6-2	4/5/2/6/3		64	{4}
group.80-1.8-0	4/5/2/5/1		384	{4}
group.80-1.8-2	4/5/2/5/5		128	{4}
group.80-1.8-4	4/5/2/5/3		128	{2}
group.80-1.8-5	4/5/2/5/6		384	{2}
group.103-1.1-0	4/32/1/2/1		288	{2, 6}
group.103-1.1-1	4/32/1/2/2		96	{2, 6}
group.109-1.1-0	4/26/1/1/1		∞	
group.141-1.1-0	4/27/2/1/1		∞	
group.142-1.1-0	4/27/3/2/1		∞	
group.142-2.1-0	4/27/3/1/1		∞	
group.143-1.1-0	4/27/4/1/1		∞	
group.144-1.1-0	4/27/1/1/1		∞	
group.163-1.1-0	4/18/4/2/1		32	{4, 8}
group.163-1.1-4	4/18/4/2/6		16	{4, 8}
group.163-1.1-6	4/18/4/2/3		32	{4, 8}
group.163-1.2-0	4/18/4/5/1		32	{4, 8}
group.163-1.2-2	4/18/4/5/3		32	{4, 8}

Table 10.1: Crystallographic groups of dimension at most 4 that do not have the R_∞-property (we omit ∞ from the spectra)

CARAT	BBNWZ	IT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.163-1.2-6	4/18/4/5/6		32	$\{4, 8\}$
group.163-1.2-7	4/18/4/5/5		32	$\{4, 8\}$
group.169-1.1-0	4/18/1/2/1		64	$\{4, 8\}$
group.169-1.1-2	4/18/1/2/3		64	$\{4, 8\}$
group.169-1.2-0	4/18/1/3/1		64	$\{4, 8\}$
group.169-1.2-1	4/18/1/3/2		64	$\{4, 8\}$
group.170-1.1-0	4/11/1/1/1		∞	
group.171-1.1-0	4/11/2/1/1		∞	
group.172-2.1-0	4/17/2/1/1		∞	
group.172-1.1-0	4/17/2/2/1		∞	
group.173-1.1-0	4/17/1/3/1		∞	
group.173-2.1-0	4/17/1/1/1		∞	
group.173-3.1-0	4/17/1/2/1		∞	
group.179-1.1-0	4/8/1/2/1		∞	$4\mathbb{N}$
group.179-1.1-1*	4/8/1/2/2		∞	$6\mathbb{N}$
group.179-1.2-0	4/8/1/1/1		∞	
group.179-1.2-1*	4/8/1/1/2		∞	$6\mathbb{N}$
group.182-1.1-0	4/9/2/1/1		∞	$8\mathbb{N} \cup \{12\}$

Table 10.1: Crystallographic groups of dimension at most 4 that do not have the R_∞-property (we omit ∞ from the spectra)

10.2 Almost-crystallographic groups

In this section, we will calculate the Reidemeister spectra of the almost-crystallographic groups of dimension at most 3 and of the almost-Bieberbach groups of dimension at most 4. The results obtained in this section were published in [DTV19] and [Ter19]. We will use the same presentations as in section 9.2.1.

10.2.1 Dimension 3

Family min.2-1.1-0. This family consists of the finitely generated, torsion-free, nilpotent groups of nilpotency class 2 and rank 3. We have already determined in theorem 5.2.2 that these groups have Reidemeister spectrum $\mathrm{Spec}_R(\Gamma) = 2\mathbb{N} \cup \{\infty\}$.

Family group.1-1.1-0. We first calculate an explicit formula for the Reidemeister number of a given automorphism.

Proposition 10.2.1. *Let* Γ *be a* 3*-dimensional almost-crystallographic group in the family group.1-1.1-0, and let* $\varphi = \xi_{(d,D)} \in \operatorname{Aut}(\Gamma)$*. Then*

$$R(\varphi) = \left(\frac{1}{\#F} \sum_{A \in F} |\det(\mathbb{1}_3 - A_* D_*)|_\infty \right) + 2S,$$

where $S \in \{0,1,2,3,4\}$ *depends on* D*,* d*, and the parameters* k_1*,* k_2*,* k_3 *and* k_4 *of* Γ*.*

Proof. The formula holds trivially for automorphisms with infinite Reidemeister number due to theorem 4.2.5. So let φ be an automorphism with finite Reidemeister number $R(\varphi)$. Under the representation λ, this automorphism will correspond to a matrix $\delta \in \operatorname{Aff}(\mathbb{R}^3)$ such that

$$\lambda(\varphi(\gamma)) = \delta \lambda(\gamma) \delta^{-1}.$$

for all $\gamma \in \Gamma$. Note that φ induces an automorphism $\varphi' = \xi_{(d/2,M)}$ on $\Gamma' := \Gamma/\langle e_1 \rangle$. Since we assumed that $R(\varphi) < \infty$, proposition 9.2.2 gives us that

$$\varphi(e_1) = e_1^{\det(M)} = e_1^{-1}.$$

Thus, δ, M and d must be of the form

$$\delta = \begin{pmatrix} -1 & n_1 & n_2 & 0 \\ 0 & m_1 & m_3 & d_1/2 \\ 0 & m_2 & m_4 & d_2/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; M = \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

where all m_i and d_j are integers, $m_1 m_4 - m_2 m_3 = -1$ and $n_1, n_2 \in \mathbb{R}$. Using a computer, one can calculate the (unique) values of n_1, n_2 and l_1, l_2, l_3 such that

$$\delta \lambda(e_2) \delta^{-1} = \lambda(e_1)^{l_1} \lambda(e_2)^{m_1} \lambda(e_3)^{m_2},$$

$$\delta \lambda(e_3) \delta^{-1} = \lambda(e_1)^{l_2} \lambda(e_2)^{m_3} \lambda(e_3)^{m_4},$$

$$\delta \lambda(\alpha) \delta^{-1} = \lambda(e_1)^{l_3} \lambda(e_2)^{d_1} \lambda(e_3)^{d_2} \lambda(\alpha).$$

From the obtained values of l_1, l_2 and l_3, we get

$$\varphi(e_1) = e_1^{-1},$$

$$\varphi(e_2) = e_1^{\frac{k_1}{2}(m_1m_2+m_1d_2-m_2d_1)-\frac{k_2}{2}(m_1+1)-\frac{k_3}{2}m_2} e_2^{m_1} e_3^{m_2},$$

$$\varphi(e_3) = e_1^{\frac{k_1}{2}(m_3m_4+m_3d_2-m_4d_1)-\frac{k_2}{2}m_3-\frac{k_3}{2}(m_4+1)} e_2^{m_3} e_3^{m_4},$$

$$\varphi(\alpha) = e_1^{\frac{k_1}{2}d_1d_2-\frac{k_2}{2}d_1-\frac{k_3}{2}d_2-k_4} e_2^{d_1} e_3^{d_2}\alpha,$$

where all exponents must be integers. This places four conditions on the m_i and d_j:

(a) $k_1(m_1m_2 + m_1d_2 - m_2d_1) - k_2(m_1 + 1) - k_3m_2 \equiv 0 \bmod 2$,

(b) $k_1(m_3m_4 + m_3d_2 - m_4d_1) - k_2m_3 - k_3(m_4 + 1) \equiv 0 \bmod 2$,

(c) $k_1d_1d_2 - k_2d_1 - k_3d_2 \equiv 0 \bmod 2$,

(d) $m_1m_4 - m_2m_3 = -1$.

We will determine $R(\varphi)$ in a very similar way to the proof of proposition 10.1.6. Let $[x]_\varphi$ be a Reidemeister class of Γ, then for any $k \in \mathbb{Z}$,

$$x = (e_1^{-k})xe_1^{2k}\varphi(e_1^{-k})^{-1},$$

therefore $x \sim_\varphi xe_1^{2k}$ for all $k \in \mathbb{Z}$. Consider the quotient group $\Gamma' = \Gamma/\langle e_1\rangle$ and let $\varphi' = \xi_{(d/2,M)}$ be the induced automorphism on this quotient. Since we assumed that $R(\varphi) < \infty$, we have that $R(\varphi') < \infty$ as well. A Reidemeister class $[x\langle e_1\rangle]_{\varphi'}$ of Γ' will lift to at most 2 Reidemeister classes of Γ: $[x]_\varphi$ and $[xe_1]_\varphi$; so the number of lifts is either 2 (when $x \not\sim_\varphi xe_1$) or 1 (when $x \sim_\varphi xe_1$). The latter happens if and only if

$$\exists z \in \Gamma : xe_1 = zx\varphi(z)^{-1}. \tag{10.8}$$

Projecting this to the quotient Γ', we have

$$\exists z \in \Gamma : x\langle e_1\rangle = zx\varphi(z)^{-1}\langle e_1\rangle. \tag{10.9}$$

Since e_1 is central in Γ and x appears exactly once on each side of the equality sign in (10.8), the e_1-component of x does not matter. Set $x = e_2^{x_2}e_3^{x_3}\alpha^{\epsilon_x}$ and $z = e_1^{z_1}e_2^{z_2}e_3^{z_3}\alpha^{\epsilon_z}$. Let us first assume that $\epsilon_z = 0$, then (10.9) is equivalent to

$$\exists z_2, z_3 \in \mathbb{Z} : (\mathbb{1}_2 - A'M)\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = 0,$$

with A' the holonomy part of $x\langle e_1\rangle$. As $R(\varphi') < \infty$, we must have $z_2 = z_3 = 0$. But then $z = e_1^{z_1}$, and (10.8) then becomes $xe_1 = xe_1^{2z_1}$. As z_1 is an integer, this is impossible. So, let us assume that $\epsilon_z = 1$. Writing out (10.8) component-wise, we find that this condition is equivalent to the following:

There exist $z_1, z_2, z_3 \in \mathbb{Z}$ such that:

(i) $2\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = (\mathbb{1}_2 - (-1)^{\epsilon_x} M)\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} - (-1)^{\epsilon_x} d,$

(ii) $k_1 z_2 z_3 - k_2 z_2 - k_3 z_3 - k_4 + 1 = 2z_1.$

Condition (i) is independent of the e_1-components, and hence can be interpreted in terms of the quotient group Γ'. In the proof of lemma 7.1.5 it was shown that, for a fixed value of ϵ_x, the number of Reidemeister classes $[x\langle e_1\rangle]_{\varphi'}$ for which a pair (z_2, z_3) satisfying (i) exists is exactly $O(\mathbb{1}_2 - M, d)$, i.e. the number of solutions $(\bar{z}_2, \bar{z}_3) \in \mathbb{Z}_2^2$ of the linear system of equations

$$\text{(i')} \quad \left(\overline{\mathbb{1}_2 - M}\right)\begin{pmatrix} \bar{z}_2 \\ \bar{z}_3 \end{pmatrix} = \bar{d}.$$

Note that the above equation is exactly condition (i) taken modulo 2.

Since ϵ_x can take two values (1 and -1), there are in total $2O(\mathbb{1}_2 - M, d)$ Reidemeister classes $[x\langle e_1\rangle]_{\varphi'}$ satisfying condition (i). On the other hand, there are $|\operatorname{tr}(M)| - O(\mathbb{1}_2 - M, d)$ Reidemeister classes of Γ' for which condition (i) does not hold.

Recall that the variable z_1 appears only in condition (ii). If we have a Reidemeister class $[x\langle e_1\rangle]_{\varphi'}$ and a pair (z_2, z_3) for which (i) holds, then we can find a $z_1 \in \mathbb{Z}$ to make condition (ii) hold if and only if

$$\text{(ii')} \quad \bar{k}_1 \bar{z}_2 \bar{z}_3 - \bar{k}_2 \bar{z}_2 - \bar{k}_3 \bar{z}_3 - \bar{k}_4 + \bar{1} = \bar{0},$$

which is exactly condition (ii) taken modulo 2.

We partition the solutions of (i') into those that do not satisfy condition (ii') and those that do. Let S be the number of the former and T the number of the latter, then $S + T = O(\mathbb{1}_2 - M, d)$. Of the $2O(\mathbb{1}_2 - M, d)$ Reidemeister classes $[x\langle e_1\rangle]_{\varphi'}$ satisfying condition (i), $2S$ lift to two distinct Reidemeister classes $[x]_\varphi$ and $[xe_1]_\varphi$, and $2T$ lift to a single Reidemeister class $[x]_\varphi$. All together, we have

$$\begin{aligned} R(\varphi) &= 2(|\operatorname{tr}(M)| - S - T) + 2(2S) + 2T \\ &= 2|\operatorname{tr}(M)| + 2S. \end{aligned}$$

If D is the matrix associated to the automorphism φ, then D_* is of the form

$$D_* = \begin{pmatrix} -1 & * & * \\ 0 & m_1 & m_3 \\ 0 & m_2 & m_4 \end{pmatrix}.$$

For any $A \in F$, let $A' = \pm\mathbb{1}'_2$ be the corresponding matrix in F'. We then have that

$$|\det(\mathbb{1}_3 - A_*D_*)| = 2|\det(\mathbb{1}_2 - A'M)| = 2|\operatorname{tr}(M)|,$$

therefore we indeed obtain the formula

$$R(\varphi) = \left(\frac{1}{\#F} \sum_{A \in F} |\det(\mathbb{1}_3 - A_*D_*)|_\infty \right) + 2S,$$

where $0 \le S \le O(\mathbb{1}_2 - M, d) \le 4$. □

Theorem 10.2.2. *Let Γ be a 3-dimensional almost-crystallographic group in the family group.1-1.1-0, with parameters k_1, k_2, k_3, k_4. Then the Reidemeister spectrum of Γ is*

- $4\mathbb{N} \cup \{\infty\}$, *if* $\bar{k}_1 = 0$ *and* $(\bar{k}_2, \bar{k}_3, \bar{k}_4) \neq (0, 0, 1)$,
- $2\mathbb{N} \cup \{\infty\}$, *if* $\bar{k}_1 = 0$ *and* $(\bar{k}_2, \bar{k}_3, \bar{k}_4) = (0, 0, 1)$,
- $4\mathbb{N} - 2 \cup \{\infty\}$, *if* $\bar{k}_1 = 1$ *and* $\bar{k}_2\bar{k}_3 + \bar{k}_4 = 0$,
- $2\mathbb{N} + 2 \cup \{\infty\}$, *if* $\bar{k}_1 = 1$ *and* $\bar{k}_2\bar{k}_3 + \bar{k}_4 = 1$,

where the bar-notation stands for the projection to $\mathbb{Z}_2$.

Proof. From the proof of proposition 10.2.1, we get that $R(\varphi) \in 2\mathbb{N}$. Taking the parity of $\operatorname{tr}(M)$ into account, we can further determine the possible Reidemeister numbers:

$$R(\varphi) \in \begin{cases} 4\mathbb{N} + 2S & \text{if } \operatorname{tr}(M) \equiv 0 \bmod 2, \\ 4\mathbb{N} + 2S - 2 & \text{if } \operatorname{tr}(M) \equiv 1 \bmod 2, \end{cases}$$

where

$$S \le O(\mathbb{1}_2 - M, d) \le \begin{cases} 4 & \text{if } \operatorname{tr}(M) \equiv 0 \bmod 2, \\ 1 & \text{if } \operatorname{tr}(M) \equiv 1 \bmod 2. \end{cases}$$

There is one special case, however. If $M \equiv \mathbb{1}_2 \bmod 2$ all entries of $\mathbb{1}_2 - M$ will be multiples of 2; so $|\det(\mathbb{1}_2 - M)| = |\operatorname{tr}(M)| \in 4\mathbb{N}$ and therefore $R(\varphi) \in 8\mathbb{N} + 2S$.

For a fixed group Γ in this family (i.e. a fixed tuple of parameters (k_1, k_2, k_3, k_4)), an automorphism $\varphi \in \operatorname{Aut}(\Gamma)$ is uniquely determined by the matrix $M \in \operatorname{GL}_2(\mathbb{Z})$

and the vector $d \in \mathbb{Z}^2$. Our goal is to find out, for each group in the family (or equivalently, for each tuple (k_1, k_2, k_3, k_4)), which M and d satisfy conditions (a) - (d) and thus produce an automorphism.

Conditions (a) - (c) are actually conditions over $\mathbb{Z}_2$, and none of the parameters k_i appear in condition (d). Therefore, only the parity of the k_i will play a role, so we need to check 16 cases, each corresponding to an element of $\mathbb{Z}_2^4$. Furthermore, a group with parameters (k_1, k_2, k_3, k_4) is isomorphic to the group with parameters $(-k_1, k_3, k_2, k_4)$, which allows us to omit the cases $(0, 1, 0, 0)$, $(0, 1, 0, 1)$, $(1, 1, 0, 0)$ and $(1, 1, 0, 1)$, leaving only 12 cases. Rather than trying to find all couples (M, d) (of which there are likely to be infinitely many), we can start by finding all couples $(\bar{M}, \bar{d}) \in \mathrm{GL}_2(\mathbb{Z}_2) \times \mathbb{Z}_2^2$ satisfying conditions (a)-(c).

The function MakeList defined in algorithm 10 does exactly this. Moreover, it assigns to every couple a set R, which is the set of possible Reidemeister numbers the corresponding automorphisms can have. The results can be found in tables B.13 to B.24. The Reidemeister spectrum of a group is a subset of (or the entirety of) the union of all these sets R.

Next, for each quadruplet of parameters, we found a family of automorphisms whose Reidemeister numbers produce the union of these sets R. These automorphisms and their Reidemeister numbers, for all (k_1, k_2, k_3, k_4), can be found in table 10.2. For the sake of brevity, we omitted ∞ from the spectra in this table. Note that all almost-Bieberbach groups belonging to this family have parameters with parities $(0, 0, 0, 1)$ and therefore have spectrum $2\mathbb{N} \cup \{\infty\}$. □

10.2.2 Dimension 4, almost-Bieberbach groups

We already determined in section 9.2.2 which families of four-dimensional almost-crystallographic groups do not have the R_∞-property. In [Dek96] it is determined which groups among these families are almost-Bieberbach groups. We use the presentations from section 9.2.2.

Family min.6-1.1-0. Every group in this family is a finitely generated, torsion-free, nilpotent group of rank 4 and nilpotency class 2. In theorem 5.2.3 it was shown that the Reidemeister spectrum of such group is always $4\mathbb{N} \cup \{\infty\}$.

Family min.7-1.1-0. The almost-Bieberbach groups in this family are those with parameters $(k_1, k_2, k_3, k_4) = (2k, 0, 0, 1)$ for some $k \in \mathbb{N}$, i.e. every almost-

Algorithm 10 Determining automorphisms and Reidemeister spectra of 3-dimensional almost-crystallographic groups in family group.1-1.1-0

```
function MakeList(k_1, k_2, k_3, k_4)
    AutList ← ∅
    for M̄ ∈ GL_2(Z_2), d̄ ∈ Z_2^2 do
        if conditions (a), (b), (c) are met then
            S ← 0
            for z̄ ∈ Z_2^2 do
                if z̄ satisfies (i') but not (ii') then
                    S ← S + 1
                end if
            end for
            if tr(M) ≡ 0 mod 2 then
                if M ≡ 𝟙_2 mod 2 then
                    R ← 8N + 2S
                else
                    R ← 4N + 2S
                end if
            else
                R ← 4N + 2S − 2
            end if
            AutList ← AutList ∪ {(M̄, d̄, R)}
        end if
    end for
    return AutList
end function
```

Bieberbach group in this family has a presentation of the form

$$\left\langle e_1, e_2, e_3, e_4, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_2 \alpha \\ [e_4, e_1] = 1 & \alpha e_3 = e_3^{-1} \alpha \\ [e_3, e_2] = 1 & \alpha e_4 = e_4^{-1} \alpha \\ [e_4, e_2] = 1 & \alpha^2 = e_1 \\ [e_4, e_3] = e_1^{2k} & \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4)$	M	d	$R(\varphi)$	$\mathrm{Spec}_R(\Gamma)$
$(0,0,0,0)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2m \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$	$4m$	$4\mathbb{N}$
$(0,0,0,1)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & m \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$	$2m$	$2\mathbb{N}$
$(0,0,1,0)$	$\left(\begin{smallmatrix} 1 & 1 \\ 2m & 2m-1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$	$4m$	$4\mathbb{N}$
$(0,0,1,1)$	$\left(\begin{smallmatrix} 1 & 1 \\ 2m & 2m-1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$	$4m$	$4\mathbb{N}$
$(0,1,1,0)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2m \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$	$4m$	$4\mathbb{N}$
$(0,1,1,1)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2m \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$	$4m$	$4\mathbb{N}$
$(1,0,0,0)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2m-1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$	$4m-2$	$4\mathbb{N}-2$
$(1,0,0,1)$	$\left(\begin{smallmatrix} 1 & 1 \\ m & m-1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$	$2m+2$	$2\mathbb{N}+2$
$(1,0,1,0)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2m-1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$	$4m-2$	$4\mathbb{N}-2$
$(1,0,1,1)$	$\left(\begin{smallmatrix} m & 1 \\ 1 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$	$2m+2$	$2\mathbb{N}+2$
$(1,1,1,0)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & m \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$	$2m+2$	$2\mathbb{N}+2$
$(1,1,1,1)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2m-1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$	$4m-2$	$4\mathbb{N}-2$

Table 10.2: Automorphisms and Reidemeister spectra for all (k_1, k_2, k_3, k_4) (we omit ∞ from the spectra)

Theorem 10.2.3. *Let* Γ *be a* 4*-dimensional almost-Bieberbach group in the family min.7-1.1-0, with parameters* $2k, 0, 0, 1$*. Then the Reidemeister spectrum of* Γ *is* $4\mathbb{N} \cup \{\infty\}$*.*

Proof. Let φ be an automorphism with finite Reidemeister number $R(\varphi)$. Under the representation λ, this automorphism will correspond to a matrix $\delta \in \mathrm{Aff}(\mathbb{R}^4)$ such that

$$\lambda(\varphi(\gamma)) = \delta\lambda(\gamma)\delta^{-1}.$$

for all $\gamma \in \Gamma$. Note that φ induces an automorphism φ' on $\Gamma' := \Gamma/\langle e_1 \rangle$ and also an automorphism $\varphi'' = \xi_{(d/2,M)}$ on $\Gamma'' := \Gamma/Z(\Gamma)$, which is the crystallographic group group.1-1.1-0. Since we assumed that $R(\varphi) < \infty$, proposition 9.2.2 gives us that

$$\varphi(e_1) = e_1^{\det(M)} = e_1^{-1}.$$

Thus, δ, M and d must be of the form

$$\delta = \begin{pmatrix} -1 & n_1 & n_2 & n_3 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & m_1 & m_3 & d_1/2 \\ 0 & 0 & m_2 & m_4 & d_2/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; M = \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

where all m_i and d_j are integers, $m_1m_4 - m_2m_3 = -1$ and $n_1, n_2 \in \mathbb{R}$. Using a computer, one can calculate the (unique) values of n_1, n_2, n_3 and l_1, l_2, l_3, l_4

such that

$$\delta\lambda(e_2)\delta^{-1} = \lambda(e_1)^{l_1}\lambda(e_2)^{-1},$$

$$\delta\lambda(e_3)\delta^{-1} = \lambda(e_1)^{l_2}\lambda(e_3)^{m_1}\lambda(e_4)^{m_2},$$

$$\delta\lambda(e_4)\delta^{-1} = \lambda(e_1)^{l_3}\lambda(e_3)^{m_3}\lambda(e_4)^{m_4},$$

$$\delta\lambda(\alpha)\delta^{-1} = \lambda(e_1)^{l_4}\lambda(e_3)^{d_1}\lambda(e_4)^{d_2}\lambda(\alpha).$$

From the obtained values of l_1, l_2, l_3 and l_4, we get

$$\varphi(e_1) = e_1^{-1},$$

$$\varphi(e_2) = e_1^{l}e_2^{-1},$$

$$\varphi(e_3) = e_1^{k(m_1m_2+m_1d_2-m_2d_1)}e_3^{m_1}e_4^{m_2},$$

$$\varphi(e_4) = e_1^{k(m_3m_4+m_3d_2-m_4d_1)}e_3^{m_3}e_4^{m_4},$$

$$\varphi(\alpha) = e_1^{kd_1d_2-1}e_3^{d_1}e_4^{d_2}\alpha,$$

with m_i, d_j, $l \in \mathbb{Z}$ and $m_1m_4 - m_2m_3 = -1$. Then D_* is of the form

$$D_* = \begin{pmatrix} -1 & * & * & * \\ 0 & -1 & * & * \\ 0 & 0 & m_1 & m_3 \\ 0 & 0 & m_2 & m_4 \end{pmatrix}.$$

Using theorem 4.2.6, we find that $R(\varphi) = 4|m_1 + m_4| \in 4\mathbb{N}$. Now, consider the family of automorphisms φ_m given by

$$\varphi_m(e_1) = e_1^{-1}, \qquad \varphi_m(e_4) = e_1^{km}e_3e_4^{m},$$

$$\varphi_m(e_2) = e_2^{-1}, \qquad \varphi_m(\alpha) = e_1^{-1}\alpha,$$

$$\varphi_m(e_3) = e_4,$$

with $m \in \mathbb{N}$. Then $R(\varphi_m) = 4m$ and hence $\mathrm{Spec}_R(\Gamma) = 4\mathbb{N} \cup \{\infty\}$. □

Family min.7-1.1-1. The almost-Bieberbach groups in this family are (up to isomorphism) those where either $(k_1, k_2, k_3, k_4) = (k, 0, 0, 0)$ with $k \in \mathbb{N}$ or $(k_1, k_2, k_3, k_4) = (2k, 1, 0, 0)$ with $k \in \mathbb{N}$. In the former case, such almost-Bieberbach group can be seen as an internal semidirect product $H_k \rtimes \mathbb{Z}$, where

$H_k = \langle e_1, e_3, e_4\rangle$ and $\mathbb{Z} = \langle\alpha\rangle$. Similarly, in the latter case, such group is an internal semidirect product $H_{2k} \rtimes \mathbb{Z}$.

The almost-Bieberbach groups with parameters $k, 0, 0, 0$ have a presentation of the form

$$\left\langle e_1, e_2, e_3, e_4, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1\alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_2\alpha \\ [e_4, e_1] = 1 & \alpha e_3 = e_3^{-1}\alpha \\ [e_3, e_2] = 1 & \alpha e_4 = e_4^{-1}\alpha \\ [e_4, e_2] = 1 & \alpha^2 = e_2 \\ [e_4, e_3] = e_1^k & \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 10.2.4. *Let* Γ *be a* 4*-dimensional almost-Bieberbach group in the family min.7-1.1-1, with parameters* $k, 0, 0, 0$*. Then the Reidemeister spectrum of* Γ *is* $4\mathbb{N} \cup \{\infty\}$*.*

Proof. Let φ be an automorphism with finite Reidemeister number $R(\varphi)$. Under the representation λ, this automorphism will correspond to a matrix $\delta \in \mathrm{Aff}(\mathbb{R}^4)$ such that

$$\lambda(\varphi(\gamma)) = \delta\lambda(\gamma)\delta^{-1}.$$

for all $\gamma \in \Gamma$. Note that φ induces an automorphism φ' on $\Gamma' := \Gamma/\langle e_1\rangle$ and also an automorphism $\varphi'' = \xi_{(d/2,M)}$ on $\Gamma'' := \Gamma/Z(\Gamma)$, which is the crystallographic group group.1-1.1-0. Since we assumed that $R(\varphi) < \infty$, proposition 9.2.2 gives us that

$$\varphi(e_1) = e_1^{\det(M)} = e_1^{-1}.$$

Thus, δ, M and d must be of the form

$$\delta = \begin{pmatrix} -1 & n_1 & n_2 & n_3 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & m_1 & m_3 & d_1/2 \\ 0 & 0 & m_2 & m_4 & d_2/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; M = \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

where all m_i and d_j are integers, $m_1m_4 - m_2m_3 = -1$ and $n_1, n_2 \in \mathbb{R}$. Using a computer, one can calculate the (unique) values of n_1, n_2, n_3 and l_1, l_2, l_3, l_4

such that

$$\delta\lambda(e_2)\delta^{-1} = \lambda(e_1)^{l_1}\lambda(e_2)^{-1},$$

$$\delta\lambda(e_3)\delta^{-1} = \lambda(e_1)^{l_2}\lambda(e_3)^{m_1}\lambda(e_4)^{m_2},$$

$$\delta\lambda(e_4)\delta^{-1} = \lambda(e_1)^{l_3}\lambda(e_3)^{m_3}\lambda(e_4)^{m_4},$$

$$\delta\lambda(\alpha)\delta^{-1} = \lambda(e_1)^{l_4}\lambda(e_2)^{-1}\lambda(e_3)^{d_1}\lambda(e_4)^{d_2}\lambda(\alpha).$$

From the obtained values of l_1, l_2, l_3 and l_4, we get

$$\varphi(e_1) = e_1^{-1},$$

$$\varphi(e_2) = e_1^{l}e_2^{-1},$$

$$\varphi(e_3) = e_1^{\frac{k}{2}(m_1m_2+m_1d_2-m_2d_1)}e_3^{m_1}e_4^{m_2},$$

$$\varphi(e_4) = e_1^{\frac{k}{2}(m_3m_4+m_3d_2-m_4d_1)}e_3^{m_3}e_4^{m_4},$$

$$\varphi(\alpha) = e_1^{\frac{1}{2}(kd_1d_2+l)}e_2^{-1}e_3^{d_1}e_4^{d_2}\alpha,$$

with m_i, d_j, $l \in \mathbb{Z}$ and $m_1m_4 - m_2m_3 = -1$, and of course all coefficients must be integers as well. Then D_* is of the form

$$D_* = \begin{pmatrix} -1 & * & * & * \\ 0 & -1 & * & * \\ 0 & 0 & m_1 & m_3 \\ 0 & 0 & m_2 & m_4 \end{pmatrix}.$$

Using theorem 4.2.6, we find that $R(\varphi) = 4|m_1 + m_4| \in 4\mathbb{N}$. Now, consider the family of automorphisms φ_m given by

$$\varphi_m(e_1) = e_1^{-1}, \qquad \varphi_m(e_4) = e_1^{km}e_3e_4^{m},$$

$$\varphi_m(e_2) = e_2^{-1}, \qquad \varphi_m(\alpha) = e_2^{-1}e_4^{m}\alpha,$$

$$\varphi_m(e_3) = e_4,$$

with $m \in \mathbb{N}$. Then $R(\varphi_m) = 4m$ and hence $\mathrm{Spec}_R(\Gamma) = 4\mathbb{N} \cup \{\infty\}$. □

The almost-Bieberbach groups with parameters $2k, 1, 0, 0$ have a presentation of the form

$$\left\langle e_1, e_2, e_3, e_4, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_2 \alpha \\ [e_4, e_1] = 1 & \alpha e_3 = e_1 e_3^{-1} \alpha \\ [e_3, e_2] = 1 & \alpha e_4 = e_4^{-1} \alpha \\ [e_4, e_2] = 1 & \alpha^2 = e_2 \\ [e_4, e_3] = e_1^{2k} & \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 10.2.5. *Let Γ be a 4-dimensional almost-Bieberbach group in the family min.7-1.1-1, with parameters $2k, 1, 0, 0$. Then the Reidemeister spectrum of Γ is $8\mathbb{N} \cup \{\infty\}$.*

Proof. Let φ be an automorphism with finite Reidemeister number $R(\varphi)$. Under the representation λ, this automorphism will correspond to a matrix $\delta \in \mathrm{Aff}(\mathbb{R}^4)$ such that

$$\lambda(\varphi(\gamma)) = \delta \lambda(\gamma) \delta^{-1}.$$

for all $\gamma \in \Gamma$. Note that φ induces an automorphism φ' on $\Gamma' := \Gamma/\langle e_1 \rangle$ and also an automorphism $\varphi'' = \xi_{(d/2,M)}$ on $\Gamma'' := \Gamma/Z(\Gamma)$, which is the crystallographic group group.1-1.1-0. Since we assumed that $R(\varphi) < \infty$, proposition 9.2.2 gives us that

$$\varphi(e_1) = e_1^{\det(M)} = e_1^{-1}.$$

Thus, δ, M and d must be of the form

$$\delta = \begin{pmatrix} -1 & n_1 & n_2 & n_3 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & m_1 & m_3 & d_1/2 \\ 0 & 0 & m_2 & m_4 & d_2/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; M = \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

where all m_i and d_j are integers, $m_1 m_4 - m_2 m_3 = -1$ and $n_1, n_2 \in \mathbb{R}$. Using a computer, one can calculate the (unique) values of n_1, n_2, n_3 and l_1, l_2, l_3, l_4

such that

$$\begin{aligned}
\delta\lambda(e_2)\delta^{-1} &= \lambda(e_1)^{l_1}\lambda(e_2)^{-1},\\
\delta\lambda(e_3)\delta^{-1} &= \lambda(e_1)^{l_2}\lambda(e_3)^{m_1}\lambda(e_4)^{m_2},\\
\delta\lambda(e_4)\delta^{-1} &= \lambda(e_1)^{l_3}\lambda(e_3)^{m_3}\lambda(e_4)^{m_4},\\
\delta\lambda(\alpha)\delta^{-1} &= \lambda(e_1)^{l_4}\lambda(e_2)^{-1}\lambda(e_3)^{d_1}\lambda(e_4)^{d_2}\lambda(\alpha).
\end{aligned}$$

From the obtained values of l_1, l_2, l_3 and l_4, we get

$$\begin{aligned}
\varphi(e_1) &= e_1^{-1},\\
\varphi(e_2) &= e_1^{l}e_2^{-1},\\
\varphi(e_3) &= e_1^{k(m_1m_2+m_1d_2-m_2d_1)-\frac{m_1+1}{2}}e_3^{m_1}e_4^{m_2},\\
\varphi(e_4) &= e_1^{k(m_3m_4+m_3d_2-m_4d_1)-\frac{m_3}{2}}e_3^{m_3}e_4^{m_4},\\
\varphi(\alpha) &= e_1^{kd_1d_2-\frac{d_1-l}{2}}e_2^{-1}e_3^{d_1}e_4^{d_2}\alpha,
\end{aligned}$$

with m_i, d_j, $l \in \mathbb{Z}$ and $m_1m_4 - m_2m_3 = -1$, and of course all coefficients must be integers as well. This forces m_1 to be odd and m_3, and because $\det(M) = -1$ we then also require m_4 to be odd. Thus, D_* is of the form

$$D_* = \begin{pmatrix} -1 & * & * & * \\ 0 & -1 & * & * \\ 0 & 0 & 2m_1' - 1 & 2m_3' \\ 0 & 0 & m_2' & 2m_4' + 1 \end{pmatrix}.$$

Using theorem 4.2.6, we find that $R(\varphi) = 8|m_1' + m_4'| \in 8\mathbb{N}$. Now, consider the family of automorphisms φ_m given by

$$\begin{aligned}
\varphi_m(e_1) &= e_1^{-1}, & \varphi_m(e_4) &= e_1^{2km-m}e_3^{2m}e_4,\\
\varphi_m(e_2) &= e_2^{-1}, & \varphi_m(\alpha) &= e_2^{-1}\alpha,\\
\varphi_m(e_3) &= e_1^{2km-k-m}e_3^{2m-1}e_4,
\end{aligned}$$

with $m \in \mathbb{N}$. Then $R(\varphi_m) = 8m$ and hence $\mathrm{Spec}_R(\Gamma) = 8\mathbb{N} \cup \{\infty\}$. □

Family min.7-1.2-0. The almost-Bieberbach groups in this family are those with parameters $(k_1, k_2, k_3, k_4) = (k, 0, 0, 1)$ for some $k \in \mathbb{N}$, i.e. every almost-

Bieberbach group in one of these families has a presentation of the form

$$\left\langle e_1, e_2, e_3, e_4, \alpha \;\middle|\; \begin{array}{ll} [e_2, e_1] = 1 & \alpha e_1 = e_1 \alpha \\ [e_3, e_1] = 1 & \alpha e_2 = e_2 \alpha \\ [e_4, e_1] = 1 & \alpha e_3 = e_2^{-1} e_3^{-1} \alpha \\ [e_3, e_2] = 1 & \alpha e_4 = e_4^{-1} \alpha \\ [e_4, e_2] = 1 & \alpha^2 = e_1 \\ [e_4, e_3] = e_1^k & \end{array} \right\rangle,$$

and the faithful representation λ is given by

$$\lambda(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 10.2.6. *Let* Γ *be a* 4*-dimensional almost-Bieberbach group in the family min.7-1.2-0, with parameters* $k, 0, 0, 1$*. Then the Reidemeister spectrum of* Γ *is* $8\mathbb{N} \cup \{\infty\}$*.*

Proof. Let φ be an automorphism with finite Reidemeister number $R(\varphi)$. Under the representation λ, this automorphism will correspond to a matrix $\delta \in \mathrm{Aff}(\mathbb{R}^4)$ such that

$$\lambda(\varphi(\gamma)) = \delta \lambda(\gamma) \delta^{-1}.$$

for all $\gamma \in \Gamma$. Note that φ induces an automorphism φ' on $\Gamma' := \Gamma/\langle e_1 \rangle$ and also an automorphism $\varphi'' = \xi_{(d/2,M)}$ on $\Gamma'' := \Gamma/Z(\Gamma)$, which is the crystallographic group group.1-1.1-0. Since we assumed that $R(\varphi) < \infty$, proposition 9.2.2 gives us that

$$\varphi(e_1) = e_1^{-1}.$$

Thus, δ, M and d must be of the form

$$\delta = \begin{pmatrix} -1 & n_1 & n_2 & n_3 & 0 \\ 0 & -1 & m_1 & m_3 & 0 \\ 0 & 0 & 2m_1 - 1 & 2m_3 & d_1 \\ 0 & 0 & m_2 & 2m_4 + 1 & d_2/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} 2m_1 - 1 & 2m_3 \\ m_2 & 2m_4 + 1 \end{pmatrix}, \qquad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

where all m_i and d_j are integers, $\det(M) = -1$ and $n_1, n_2, n_3 \in \mathbb{R}$. Using a computer, one can calculate the (unique) values of n_1, n_2, n_3 and l_1, l_2, l_3, l_4

such that

$$\delta\lambda(e_2)\delta^{-1} = \lambda(e_1)^{l_1}\lambda(e_2)^{-1},$$

$$\delta\lambda(e_3)\delta^{-1} = \lambda(e_1)^{l_2}\lambda(e_2)^{m_1}\lambda(e_3)^{2m_1-1}\lambda(e_4)^{m_2},$$

$$\delta\lambda(e_4)\delta^{-1} = \lambda(e_1)^{l_3}\lambda(e_2)^{m_3}\lambda(e_3)^{2m_3}\lambda(e_4)^{2m_4+1},$$

$$\delta\lambda(\alpha)\delta^{-1} = \lambda(e_1)^{l_4}\lambda(e_2)^{d_1}\lambda(e_3)^{2d_1}\lambda(e_4)^{d_2}\lambda(\alpha).$$

From the obtained values of l_1, l_2, l_3 and l_4, we get

$$\varphi(e_1) = e_1^{-1},$$

$$\varphi(e_2) = e_2^{-1}e_1^{k(2m_1m_2+2m_1d_2-2m_2d_1-m_2-d_2)-2l},$$

$$\varphi(e_3) = e_2^{m_1}e_3^{-1+2m_1}e_4^{m_2}e_1^{l},$$

$$\varphi(e_4) = e_2^{m_3}e_3^{2m_3}e_4^{1+2m_4}e_1^{k(2m_3m_4+m_3d_2+m_3-2m_4d_1-d_1)},$$

$$\varphi(\alpha) = e_2^{d_1}e_3^{2d_1}e_4^{d_2}e_1^{kd_1d_2-1}\alpha,$$

with m_1, m_2, m_3, m_4, d_1, d_2, $l \in \mathbb{Z}$ and $m_1 - m_4 + 2m_1m_4 - m_2m_3 = 0$. Then D_* is of the form

$$D_* = \begin{pmatrix} -1 & * & * & * \\ 0 & -1 & * & * \\ 0 & 0 & -1+2m_1 & 2m_3 \\ 0 & 0 & m_2 & 1+2m_4 \end{pmatrix}.$$

Using theorem 4.2.6, we find that $R(\varphi) = 8|m_1 + m_4| \in 8\mathbb{N}$. Now, consider the family of automorphisms φ_m given by

$$\varphi_m(e_1) = e_1^{-1}, \qquad \varphi_m(e_4) = e_1^{km}e_2^{m}e_3^{2m}e_4,$$

$$\varphi_m(e_2) = e_1^{k(2m-1)}e_2^{-1}, \qquad \varphi_m(\alpha) = e_1^{-1}\alpha,$$

$$\varphi_m(e_3) = e_2^{m}e_3^{2m-1}e_4,$$

with $m \in \mathbb{N}$. Then $R(\varphi_m) = 8m$ and hence $\mathrm{Spec}_R(\Gamma) = 8\mathbb{N} \cup \{\infty\}$. $\square$

Chapter 11

Reidemeister zeta functions

11.1 Existence

The goal of this section is to determine which almost-crystallographic groups admit Reidemeister zeta functions of automorphisms. In order to do so, it is helpful to have criteria for the (non-)existence of these functions. A first and obvious criterion would be that a group with the R_∞-property does not admit any Reidemeister zeta functions. Another criterion is the existence of a specific characteristic subgroup:

Proposition 11.1.1. *Let Γ be an almost-crystallographic group with a characteristic subgroup $H \cong \mathbb{Z}$. Then Γ does not admit any Reidemeister zeta functions of automorphisms.*

Proof. Let $x \in \Gamma$ such that $H = \langle x \rangle$. As H is normal and abelian, we must have that H is a subgroup of the translation subgroup N of Γ. Since N is nilpotent and H is normal in N, we must have that the intersection $H \cap Z(N)$ is non-trivial.

So, there exists some $k > 0$ such that $x^k \in Z(N)$. In fact, as N is torsion-free, $N/Z(N)$ is torsion-free as well and hence we have that $x \in Z(N)$, thus $H \leq Z(N)$. Let $\varphi = \xi_{(d,D)}$ be an automorphism of Γ. As $x \in Z(N)$, it then follows that $\varphi(x) = D(x)$. Either $\varphi(x) = D(x) = x$ or $\varphi(x) = D(x) = x^{-1}$; in any case we have that $D^2(x) = x$.

Let G be the Lie group that Γ is modelled on. Then there exists a non-zero element X (corresponding to $x \in G$) in the associated Lie algebra $\mathfrak{g}$ with

$D_*^2(X) = X$ and therefore $\det(\mathbb{1} - D_*^2) = 0$. So certainly $R(\varphi^2) = \infty$ and we can conclude that the Reidemeister zeta function $R_\varphi(z)$ does not exist. □

The next criterion deals with the size of the outer automorphism group.

Proposition 11.1.2. *If a crystallographic group* Γ *has finite outer automorphism group, then it has no Reidemeister zeta functions of automorphisms.*

Proof. We know from theorem 3.3.8 that if $\mathrm{Out}(\Gamma)$ is finite, then N_Γ is finite as well. Let $\varphi = \xi_{(d,D)}$ be an automorphism, then $d \in \mathbb{R}^n$ and $D \in N_\Gamma$. Since N_Γ is finite, there exists some $k \in \mathbb{N}$ such that $D^k = \mathbb{1}_n$. But then $\det(\mathbb{1}_n - D^k) = 0$, hence $R(\varphi^k) = \infty$ and thus the Reidemeister zeta function of φ does not exist. □

Similar to when we tried to determine the R_∞-property, it can be helpful to look at quotients by characteristic subgroups.

Proposition 11.1.3. *Let* Γ *be an almost-crystallographic group. If* Γ *has a characteristic subgroup* H *such that* Γ/H *does not admit Reidemeister zeta functions of automorphisms, then* Γ *does not admit them either.*

Proof. Let $\varphi \in \mathrm{Aut}(\Gamma)$. Since H is characteristic, φ induces an automorphism φ' on Γ/H. But this quotient group does not admit Reidemeister zeta functions, hence either $R(\varphi'^k) = \infty$ for some $k \in \mathbb{N}$, or the radius of convergence of $R_{\varphi'}(z)$ is zero.

First, consider the case where $R(\varphi'^k) = \infty$ for some $k \in \mathbb{N}$. Since φ'^k is the automorphism of induced by φ^k, by lemma 2.5.10(1) we have that

$$R(\varphi^k) \geq R(\varphi'^k) = \infty,$$

and thus $R(\varphi^k) = \infty$, therefore $R_\varphi(z)$ does not exist.

Second, consider the case where the radius of convergence r' of $R_{\varphi'}(z)$ is zero. Let r be the radius of convergence of $R_\varphi(z)$. Using lemma 2.5.10(1) once again, we find that

$$r^{-1} = \limsup_{k\to\infty} \sqrt[k]{\frac{R(\varphi^k)}{k}} \geq \limsup_{k\to\infty} \sqrt[k]{\frac{R(\varphi'^k)}{k}} = r'^{-1} = \infty,$$

and hence $r = 0$, therefore $R_\varphi(z)$ does not exist.

In both cases the Reidemeister zeta function $R_\varphi(z)$ does not exist, and since this holds for an arbitrary automorphism φ, Γ does not admit Reidemeister zeta functions of automorphisms. □

In low dimensions, almost-crystallographic group admitting Reidemeister zeta functions must be crystallographic.

Theorem 11.1.4. *Let Γ be a non-crystallographic, almost-crystallographic group of dimension 3 or 4. Then Γ does not admit any Reidemeister zeta functions of automorphisms.*

Proof. Let Γ be a non-crystallographic, almost-crystallographic group of dimension 3 or 4 and let N be its translation subgroup, which has nilpotency class $c \geq 2$. As shown in section 5.2, the isolator $\sqrt[N]{\gamma_c(N)}$ is isomorphic to $\mathbb{Z}$ and is characteristic in Γ. By proposition 11.1.1, this means Γ does not admit any Reidemeister zeta functions of automorphisms. □

We can even go a step further. In [Mal00] it is shown that if a finitely generated, torsion-free, nilpotent group which is not abelian admits an automorphism φ with affine homotopy lift D for which D_* has no roots of unity as eigenvalues, then the dimension of this group must be at least 6. Moreover, explicit examples of such automorphisms on groups of dimension 6 are provided. Thus, we may conclude that Reidemeister zeta functions of automorphisms on non-crystallographic almost-crystallographic groups exist only in dimension 6 and higher.

Since we are only interested in dimension 4 and lower, we may limit ourselves to crystallographic groups with infinite outer automorphism group. The following theorem proves the existence of Reidemeister zeta functions of automorphisms for many of these groups in dimension 4.

Theorem 11.1.5. *Let Γ be a crystallographic group of dimension 4 such that every matrix $A \in F$ is of the form*

$$A = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix},$$

with $A_1, A_2 \in \{-\mathbb{1}_2, \mathbb{1}_2\}$. Suppose that $\varphi = \xi_{(d,D)}$ is an automorphism of Γ with D of the form

$$D = \begin{pmatrix} M & * \\ 0 & M \end{pmatrix},$$

where $M \in \mathrm{GL}_2(\mathbb{Z})$ has eigenvalues λ, μ with $|\lambda| > 1$, $|\mu| < 1$. Then the Reidemeister zeta function $R_\varphi(z)$ exists.

Proof. We have to prove two things: that $R(\varphi^k) < \infty$ for all $k \in \mathbb{N}$, and that the radius of convergence of $R_\varphi(z)$ is non-zero.

For the former, by theorem 4.2.5 we must prove that $\det(\mathbb{1}_4 - AD^k) \neq 0$ for all $A \in F$. We have that

$$\det(\mathbb{1}_4 - AD^k) = \det\begin{pmatrix} \mathbb{1}_2 - A_1M^k & * \\ 0 & \mathbb{1}_2 - A_2M^k \end{pmatrix}$$

$$= \det(\mathbb{1}_2 - A_1M^k)\det(\mathbb{1}_2 - A_2M^k).$$

Since $A_1, A_2 \in \{-\mathbb{1}_2, \mathbb{1}_2\}$, it suffices to prove that $\det(\mathbb{1}_2 - M^k) \neq 0$ and $\det(\mathbb{1}_2 + M^k) \neq 0$, or equivalently that M^k does not have an eigenvalue equal to 1 or -1. But the eigenvalues of M^k are λ^k and μ^k, for which we know that $|\lambda^k| = |\lambda|^k > 1$ and $|\mu^k| = |\mu|^k < 1$. Therefore these determinants are indeed non-zero.

Next, recall that the radius of convergence r of $R_\varphi(z)$ is given by

$$r^{-1} = \limsup_{k\to\infty} \sqrt[k]{\frac{R(\varphi^k)}{k}},$$

hence it suffices to prove that this limit is finite. From proposition 2.5.14, we know that

$$R(\varphi^k) \leq \sum_{A\in F} |\det(\mathbb{1}_4 - AD^k)|.$$

Note that

$$|\det(\mathbb{1}_2 \pm M^k)| = |(1 \pm \lambda^k)(1 \pm \mu^k)|$$

$$= |1 \pm \lambda^k \pm \mu^k + \det(M)^k|$$

$$\leq 4|\lambda|^k,$$

and thus we have for any $A \in F$ that

$$|\det(\mathbb{1}_4 - AD^k)| = |\det(\mathbb{1}_2 - A_1M^k)||\det(\mathbb{1}_2 - A_2M^k)|$$

$$\leq (4|\lambda|^k)^2$$

$$= 16|\lambda|^{2k}.$$

For $k \geq 16 \cdot \#F$, we will have that

$$R(\varphi^k) \leq \sum_{A\in F} |\det(\mathbb{1}_4 - AD^k)|$$

$$\leq \#F \cdot 16|\lambda|^{2k}$$

$$\leq k|\lambda|^{2k},$$

and hence

$$\limsup_{k\to\infty} \sqrt[k]{\frac{R(\varphi^k)}{k}} \le \limsup_{k\to\infty} \sqrt[k]{\frac{k|\lambda|^{2k}}{k}} = |\lambda|^2,$$

thus $r \ge |\lambda|^{-2} > 0$ and hence $R_\varphi(z)$ exists. □

Below, we determine for certain (families of) crystallographic groups whether or not they admit Reidemeister zeta functions of automorphisms.

min.2-1.1-0, min.6-1.1-0, min.15-1.1-0. These groups are all isomorphic to $\mathbb{Z}^n$ for $n \in \{2, 3, 4\}$. We have already shown that these admit Reidemeister zeta functions in example 2.6.11.

min.18-1.1-0, min.18-1.1-1. These groups are given by

$$\Gamma = \langle \mathbb{Z}^4, \alpha \rangle \text{ with } \alpha = (\begin{pmatrix} \epsilon \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}),$$

where $\epsilon = 0$ corresponds to min.18-1.1-0 and $\epsilon = 1/2$ to min.18-1.1-1. The automorphism $\varphi = \xi_{(0,D)}$ with D given by

$$D = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 7 \end{pmatrix}.$$

satisfies the requirements of theorem 11.1.5, hence $R_\varphi(z)$ exists.

min.18-1.2-0, min.18-1.2-1. These groups are given by

$$\Gamma = \langle \mathbb{Z}^4, \alpha \rangle \text{ with } \alpha = (\begin{pmatrix} \epsilon \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}),$$

where $\epsilon = 0$ corresponds to min.18-1.2-0 and $\epsilon = 1/2$ to min.18-1.2-1. The automorphism $\varphi = \xi_{(0,D)}$ with D given by

$$D = \begin{pmatrix} 1 & 4 & -2 & 0 \\ 2 & 7 & -3 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 7 \end{pmatrix}$$

satisfies the requirements of theorem 11.1.5, hence $R_\varphi(z)$ exists.

min.18-1.3-0. This group is given by

$$\Gamma = \langle \mathbb{Z}^4, \alpha \rangle \text{ with } \alpha = (0, \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}).$$

The automorphism $\varphi = \xi_{(0,D)}$ with D given by

$$D = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 7 \end{pmatrix}$$

satisfies the requirements of theorem 11.1.5, hence $R_\varphi(z)$ exists.

group.1-1.1-0, group.5-1.1-0, group.26-1.1-0. These groups all have diagonal holonomy $\mathbb{Z}_2$. We have already shown that these admit Reidemeister zeta functions in theorem 7.2.20.

group.28-1.1-0, group.28-1.1-1, group.28-1.1-2. These groups are given by

$$\Gamma = \langle \mathbb{Z}^4, \alpha, \beta \rangle \text{ with } \alpha = (\begin{pmatrix} \delta \\ 0 \\ \epsilon \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}),$$

$$\beta = (\begin{pmatrix} \delta \\ 0 \\ \epsilon \\ 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}),$$

where $\delta, \epsilon = 0$ corresponds to group.28-1.1-0, $\delta = 1/2, \epsilon = 0$ to group.28-1.1-1 and $\delta, \epsilon = 1/2$ to group.28-1.1-2. The automorphism $\varphi = \xi_{(0,D)}$ with D given by

$$D = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 7 \end{pmatrix}$$

satisfies the requirements of theorem 11.1.5, hence $R_\varphi(z)$ exists.

group.28-1.2-0, group.28-1.2-1, group.28-1.2-2. These groups are given by

$$\Gamma = \langle \mathbb{Z}^4, \alpha, \beta \rangle \text{ with } \alpha = (\begin{pmatrix} \epsilon \\ 0 \\ 0 \\ \delta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}),$$

$$\beta = (\begin{pmatrix} \epsilon \\ 0 \\ 0 \\ \delta \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}),$$

where $\delta, \epsilon = 0$ corresponds to group.28-1.1-0, $\delta = 1/2, \epsilon = 0$ to group.28-1.1-1 and $\delta, \epsilon = 1/2$ to group.28-1.1-2. The automorphism $\varphi = \xi_{(0,D)}$ with D given by

$$D = \begin{pmatrix} 1 & 4 & -2 & 0 \\ 2 & 7 & -3 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 7 \end{pmatrix}$$

satisfies the requirements of theorem 11.1.5, hence $R_\varphi(z)$ exists.

group.28-1.3-0. This group is given by

$$\Gamma = \langle \mathbb{Z}^4, \alpha, \beta \rangle \text{ with } \alpha = (0, \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}),$$

$$\beta = (0, \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}).$$

The automorphism $\varphi = \xi_{(0,D)}$ with D given by

$$D = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 7 \end{pmatrix}$$

satisfies the requirements of theorem 11.1.5, hence $R_\varphi(z)$ exists.

group.179-1.1-0, group.179-1.1-1, group.179-1.2-0, group.179-1.2-1. If Γ is any of these groups, then $\Gamma/Z(\Gamma)$ is the crystallographic group min.5-1.1-0. This group has finite outer automorphism group and therefore does not admit any Reidemeister zeta functions of automorphisms. By proposition 11.1.3, Γ then does not admit any Reidemeister zeta functions of automorphisms either.

group.182-1.1-0. This group is the direct product $\Gamma_1 \times \Gamma_2$, where Γ_1 is the group group.1-1.1-0 and Γ_2 is min.5-1.1-0. Since both factors are characteristic, we have that $\Gamma/(\Gamma_1 \times 1)$ is min.5-1.1-0. By the same reasoning as for the previous four groups, Γ does not admit any Reidemeister zeta functions of automorphisms.

We summarise these results in table 11.1 below. This table contains the results mentioned above, as well as the groups that have rank 1 centre, which do not admit Reidemeister zeta functions of automorphisms by proposition 11.1.3.

CARAT	BBNWZ	IT	rank($Z(\Gamma)$)	admits $R_\varphi(z)$?
min.2-1.1-0*	2/1/1/1/1	2/1	2	yes
min.6-1.1-0*	3/1/1/1/1	3/1	3	yes
min.7-1.1-0	3/2/1/1/1	3/3	1	no
min.7-1.1-1*	3/2/1/1/2	3/4	1	no
min.7-1.2-0	3/2/1/2/1	3/5	1	no
min.15-1.1-0*	4/1/1/1/1		4	yes
min.17-1.1-0	4/2/2/1/1		1	no
min.17-1.1-1*	4/2/2/1/2		1	no
min.17-1.2-0	4/2/2/2/1		1	no
min.18-1.1-0	4/3/1/1/1		2	yes
min.18-1.1-1*	4/3/1/1/2		2	yes
min.18-1.2-0	4/3/1/2/1		2	yes
min.18-1.2-1*	4/3/1/2/2		2	yes
min.18-1.3-0	4/3/1/3/1		2	yes
min.36-1.1-0	4/10/1/1/1		0	
min.43-1.1-0	4/28/1/1/1		0	
min.44-1.1-0	4/28/2/1/1		0	
max.6-1.1-0	4/26/2/1/1		0	
max.6-1.1-1	4/26/2/1/2		0	
group.1-1.1-0	2/1/2/1/1	2/2	0	yes
group.5-1.1-0	3/1/2/1/1	3/2	0	yes
group.26-1.1-0	4/1/2/1/1		0	yes
group.28-1.1-0	4/3/2/1/1		0	yes
group.28-1.1-1	4/3/2/1/2		0	yes
group.28-1.1-2	4/3/2/1/3		0	yes

Table 11.1: Existence of Reidemeister zeta functions of automorphisms for crystallographic groups of dimension at most 4 with infinite outer automorphism group that do not have the R_∞-property

CARAT	BBNWZ	IT	rank($Z(\Gamma)$)	admits $R_\varphi(z)$?
group.28-1.2-0	4/3/2/2/1		0	yes
group.28-1.2-1	4/3/2/2/2		0	yes
group.28-1.2-2	4/3/2/2/3		0	yes
group.28-1.3-0	4/3/2/3/1		0	yes
group.96-1.1-0	4/16/1/1/1		0	
group.96-1.1-1	4/16/1/1/2		0	
group.96-2.1-0	4/16/1/2/1		0	
group.96-2.1-1	4/16/1/2/2		0	
group.96-2.1-2	4/16/1/2/3		0	
group.96-3.1-0	4/16/1/3/1		0	
group.109-1.1-0	4/26/1/1/1		0	
group.141-1.1-0	4/27/2/1/1		0	
group.142-1.1-0	4/27/3/2/1		0	
group.142-2.1-0	4/27/3/1/1		0	
group.143-1.1-0	4/27/4/1/1		0	
group.144-1.1-0	4/27/1/1/1		0	
group.170-1.1-0	4/11/1/1/1		0	
group.171-1.1-0	4/11/2/1/1		0	
group.172-2.1-0	4/17/2/1/1		0	
group.172-1.1-0	4/17/2/2/1		0	
group.173-1.1-0	4/17/1/3/1		0	
group.173-2.1-0	4/17/1/1/1		0	
group.173-3.1-0	4/17/1/2/1		0	
group.179-1.1-0	4/8/1/2/1		2	no
group.179-1.1-1*	4/8/1/2/2		2	no
group.179-1.2-0	4/8/1/1/1		2	no
group.179-1.2-1*	4/8/1/1/2		2	no
group.182-1.1-0	4/9/2/1/1		0	no

Table 11.1: Existence of Reidemeister zeta functions of automorphisms for crystallographic groups of dimension at most 4 with infinite outer automorphism group that do not have the R_∞-property

11.2 Rationality

In the previous section, we only found very few almost-crystallographic groups in dimensions 2 and 3 that admit Reidemeister zeta functions of automorphisms. In fact, these groups were always $\mathbb{Z}^n$, a crystallographic group with diagonal holonomy $\mathbb{Z}_2$, or a Bieberbach group. We know from example 2.6.7, corollary 7.2.19 and theorem 4.2.8 that Reidemeister zeta functions on these groups are rational. Thus, we may state the following result:

Theorem 11.2.1. *A Reidemeister zeta function of an almost-crystallographic group of dimension at most* 3 *is rational.*

Appendix

Appendix A

Isogredience

In this chapter we will study isogredience numbers, which are closely related to Reidemeister numbers, for almost-crystallographic groups. We refer to [FT15; LL00] for more information on isogredience numbers.

A.1 Preliminaries

Isogredience numbers have a similar topological motivation as Reidemeister numbers. Let X be a compact topological space that admits a universal cover $\tilde{X}$. A lift $\tilde{f} : \tilde{X} \to \tilde{X}$ of a self-map $f : X \to X$ induces an endomorphism f_* on the group of covering transformations $\mathcal{D}(X) \cong \pi_1(X)$, namely

$$\tilde{f} \circ \gamma = f_*(\gamma) \circ \tilde{f},$$

for all $\gamma \in \mathcal{D}(X)$. A different lift $\tilde{f}'$ will induce an endomorphism f'_* that differs from f_* by an inner automorphism. Now consider the case where f is a homeomorphism, then f_* is actually an automorphism. The set of all these induced automorphisms f_* then coincides with an element $\Phi \in \operatorname{Out}(\mathcal{D}(X))$.

Recall that two lifts $\tilde{f}_1$ and $\tilde{f}_2$ are Reidemeister equivalent if they are conjugate up to an element γ of $\mathcal{D}(X)$, i.e.

$$\tilde{f}_1 = \gamma \circ \tilde{f}_2 \circ \gamma^{-1}.$$

In terms of their induced automorphisms f_{1*}, f_{2*}, these lifts $\tilde{f}_1$ and $\tilde{f}_2$ being conjugate up to an element of $D(X)$ is equivalent to f_{1*} and f_{2*} being conjugate up to an inner automorphism:

$$f_{1*} = \iota_\gamma \circ f_{2*} \circ \iota_\gamma^{-1}.$$

Thus, the equivalence relation on the lifts induces an equivalence relation on $\Phi \in \mathrm{Out}(\mathcal{D}(X))$. It is this equivalence relation, applied to almost-crystallographic groups, we will study in this chapter.

A.1.1 Group-theoretic isogredience number

The basic definitions below are very reminiscent of those in section 2.5.1, where we defined the (group-theoretic) Reidemeister number and related concepts.

Definition A.1.1. Let G be a group and $\Phi \in \mathrm{Out}(G)$. Define an equivalence relation $\sim$ on Φ by

$$\forall \varphi_1, \varphi_2 \in \Phi : \varphi_1 \sim \varphi_2 \iff \exists \iota \in \mathrm{Inn}(G) : \varphi_1 = \iota \circ \varphi_2 \circ \iota^{-1}.$$

The equivalence classes are called *isogredience classes*, and we will denote the isogredience class of φ by $[\varphi]_\Phi$. The set of isogredience classes of Φ is denoted by $\mathfrak{S}(\Phi)$. The *isogredience number* $S(\Phi)$ is the cardinality of $\mathfrak{S}(\Phi)$ and is therefore always a positive integer or infinity.

Definition A.1.2. The *isogredience spectrum* of a group G is the set

$$\mathrm{Spec}_S(G) = \{S(\Phi) \mid \Phi \in \mathrm{Out}(G)\}.$$

If $\mathrm{Spec}_S(G) = \{\infty\}$ we say that G has the *S_∞-property*, and if $\mathrm{Spec}_S(G) = \mathbb{N} \cup \{\infty\}$ we say G has *full* isogredience spectrum.

While at first glance isogredience classes seem quite different to Reidemeister classes, since they are classes of an element $\Phi \in \mathrm{Out}(G)$ rather than classes of G itself, they are very closely related. The following lemma shows that an isogredience number can always be seen as a Reidemeister number.

Lemma A.1.3 (see [FT15, Lemma 3.3])**.** *Let G be a group and let $\varphi \in \mathrm{Aut}(G)$, $\Phi \in \mathrm{Out}(G)$ such that $\varphi \in \Phi$. Then $S(\Phi) = R(\varphi')$, where φ' is the induced automorphism by φ on $G/Z(G)$.*

Proof. For any $\psi \in \Phi$, there exists some $g \in G$ such that $\psi = \iota_g \circ \varphi$, where ι_g is the inner automorphism $h \mapsto ghg^{-1}$. This g is not unique, but defined up to multiplication by a central element – this follows from the one-to-one correspondence between $\mathrm{Inn}(G)$ and $G/Z(G)$.

Let $\varphi_1, \varphi_2 \in \Phi$, then there exist $g_1, g_2 \in G$ such that $\varphi_1 = \iota_{g_1} \circ \varphi$ and $\varphi_2 = \iota_{g_2} \circ \varphi$. Then

$$\begin{aligned}
\varphi_1 \sim \varphi_2 &\iff \exists h \in G : \varphi_1 = \iota_h \circ \varphi_2 \circ \iota_h^{-1} \\
&\iff \exists h \in G : \iota_{g_1} \circ \varphi = \iota_h \circ \iota_{g_2} \circ \varphi \circ \iota_h^{-1} \\
&\iff \exists h \in G : \iota_{g_1} \circ \varphi = \iota_{hg_2\varphi(h)^{-1}} \circ \varphi \\
&\iff \exists h \in G : \iota_{g_1} = \iota_{hg_2\varphi(h)^{-1}} \\
&\iff \exists hZ(G) \in G/Z(G) : g_1 Z(G) = hg_2\varphi(h)^{-1}Z(G) \\
&\iff g_1 Z(G) \sim_{\varphi'} g_2 Z(G).
\end{aligned}$$

This means the map

$$\mathfrak{S}(\Phi) \to \mathfrak{R}(\varphi') : [\iota_g \circ \varphi]_\Phi \mapsto [gZ(G)]_{\varphi'}$$

is a bijection, from which follows that $S(\Phi) = R(\varphi')$. □

We may exploit this relation to deduce properties of the isogredience number. For example, the following lemma is an isogredience analogue to property (1) in lemma 2.5.10.

Lemma A.1.4. *Let G be a group with characteristic subgroup N, let $\Phi \in \mathrm{Out}(G)$ and Φ' the corresponding element of $\mathrm{Out}(G/N)$. Then the map*

$$\hat{p} : \mathfrak{S}(\Phi) \to \mathfrak{S}(\Phi') : [\iota_g \circ \varphi]_\Phi \mapsto [\iota_{gN} \circ \varphi']_{\Phi'}$$

is surjective, and hence $S(\Phi) \geq S(\Phi')$.

Proof. Consider the normal subgroup H given by

$$H = \{h \in G \mid \forall g \in G : [h, g] \in N\}.$$

This group has the following properties:

(1) $N \lhd H \lhd G$,

(2) $Z(G) \lhd H \lhd G$,

(3) $\varphi(H) \subseteq H$,

(4) $H/N = Z(G/N)$.

By the third isomorphism theorem, we have that

$$\frac{G/Z(G)}{H/Z(G)} \cong \frac{G}{H} \cong \frac{G/N}{H/N} = \frac{G/N}{Z(G/N)}.$$

Let φ_Z be the automorphism on $G/Z(G)$ induced by φ, and let φ'_Z be the automorphism induced on $(G/N)/Z(G/N)$. We can then apply lemma 2.5.10 to the group $G/Z(G)$ and the normal subgroup $H/Z(G)$ to obtain the surjective map $\hat{p} : \mathfrak{R}(\varphi_Z) \to \mathfrak{R}(\varphi'_Z)$. Combining this with the bijection from lemma A.1.3 gives us the desired surjective map. □

Of course, this means we can now formulate an isogredience analogue to corollary 2.5.12.

Corollary A.1.5. *Let N be a characteristic subgroup of G. If the quotient G/N has the S_∞-property, then so does G.*

Proof. This follows directly from lemma A.1.4. □

Finally, using the above lemmas and corollary, we obtain the following relations between the Reidemeister and isogredience numbers of a group and its quotient by the centre.

Proposition A.1.6. *Let G be a group and let $\varphi \in \operatorname{Aut}(G)$, $\Phi \in \operatorname{Out}(G)$ such that $\varphi \in \Phi$. We then have that:*

(1) $R(\varphi) \geq S(\Phi)$,

(2) $\operatorname{Spec}_S(G) \subseteq \operatorname{Spec}_R(G/Z(G))$,

(3) if G has the S_∞-property, then it also has the R_∞-property,

(4) if $G/Z(G)$ has the R_∞-property, then G has S_∞-property,

(5) if $Z(G) = 1$, then $R(\varphi) = S(\Phi)$ and $\operatorname{Spec}_R(G) = \operatorname{Spec}_S(G)$.

A.2 The S_∞-property

Since we are often dealing with a quotient $G/Z(G)$, let us remark that in the case of almost-crystallographic groups, such quotient is almost-crystallographic as well.

Theorem A.2.1 (see [IM96, Proposition 3.1]). *Let Γ be an almost-crystallographic group with translation subgroup N and holonomy group F. Then $\Gamma/Z(\Gamma)$ is an almost-crystallographic group with translation subgroup $N/Z(\Gamma)$ and holonomy group F, unless $\Gamma = Z(\Gamma)$, in which case the quotient is of course trivial.*

To determine which almost-crystallographic groups (do not) have the S_∞-property, we will heavily exploit the relationship with Reidemeister numbers as given by lemma A.1.3 and proposition A.1.6. Since we already know which almost-crystallographic groups Γ have the R_∞-property, we can use this information in the following way to determine whether Γ has the S_∞-property:

- Γ does not have the R_∞-property. Then Γ does not have the S_∞-property.
- Γ has the R_∞-property.
  - $Z(\Gamma) = 1$. Then Γ has the S_∞-property.
  - $Z(\Gamma) \neq 1$.
    * $\Gamma/Z(\Gamma)$ has the R_∞-property. Then Γ has the S_∞-property.
    * $\Gamma/Z(\Gamma)$ does not have the R_∞-property. No information gained.

If no information is gained, we proceed similarly to how we determined which groups have the R_∞-property.

Let us also give the isogredience analogue to proposition 7.1.1.

Proposition A.2.2. *Let id_Γ be the identity morphism on an almost-crystallographic group Γ, and let $\Phi \in \mathrm{Out}(\Gamma)$ such that $\mathrm{id}_\Gamma \in \Phi$. If Γ is not abelian, then $S(\Phi) = \infty$, otherwise $S(\Phi) = 1$.*

Proof. Let $\Gamma' := \Gamma/Z(\Gamma)$, then id_Γ induces $\mathrm{id}_{\Gamma'}$ on Γ', and hence $S(\Phi) = R(\mathrm{id}_{\Gamma'})$. If Γ is not abelian, then $R(\mathrm{id}_{\Gamma'}) = \infty$ by proposition 7.1.1, otherwise $\Gamma/Z(\Gamma)$ is trivial and then $R(\mathrm{id}'_\Gamma) = 1$. □

A.2.1 Crystallographic groups

Just like for the R_∞-property, we can use an algorithm for the crystallographic groups with finite outer automorphism group, and have to work by hand otherwise. In the former case, we use algorithm 11, which is a slightly modified version of algorithm 2.

Algorithm 11 Determining if a crystallographic group Γ has the S_∞-property

```
function HasSinfinityProperty(Γ)
    Γ' ← Γ/Z(Γ)
    N_F ← N_{GL_n(ℤ)}(F)
    if #N_F = ∞ then
        return fail
    else
        N_{Γ'} ← ∅
        for D ∈ N_F do
            if ExtendsToAutomorphism(D, Γ) ≠ fail then
                D' ← induced automorphism by D on ℤ^n/Z(Γ)
                N_{Γ'} ← N_{Γ'} ∪ {D'}
            end if
        end for
        for D' ∈ N_{Γ'} do
            S_∞ ← false
            for A' ∈ F' do
                if det(𝟙 − A'D') = 0 then
                    S_∞ ← true
                end if
            end for
            if S_∞ = false then
                return false
            end if
        end for
        return true
    end if
end function
```

In the latter case, we will make use of the following theorem, which is the isogredience analogue to theorem 9.1.2, and can easily be proven in a similar way.

Theorem A.2.3. *Let F be the holonomy group of an n-dimensional $\mathbb{Z}$-class of crystallographic groups. If $\mathbb{Z}^n \rtimes F$ has the S_∞-property, then so does every other crystallographic group in the same $\mathbb{Z}$-class.*

Thus, if the outer automorphism group is infinite (and hence N_F is infinite), we can try two things:

1. Show that all crystallographic groups in a $\mathbb{Z}$-class with holonomy group F have the S_∞-property, by finding a characteristic subgroup N of $\mathbb{Z}^n \rtimes F$

such that $(\mathbb{Z}^n \rtimes F)/N$ has the S_∞-property. This relies on corollary A.1.5 and theorem A.2.3.

2. Show that a crystallographic group Γ does not have the S_∞-property, by checking for random matrices $D \in N_F$ whether they belong to N_Γ (using algorithm 1) and whether any automorphism $\xi_{(d,D)}$ induces an automorphism on $\Gamma/Z(\Gamma)$ with finite Reidemeister number.

We have applied the above to all crystallographic groups up to dimension 4, thereby determining the groups that do not have the S_∞-property. For the groups with finite outer automorphism group, those without the S_∞-property also did not have the R_∞-property (see tables B.1 to B.4), with a sole exception: the 3-dimensional Bieberbach group min.13-1.1-1, which has the R_∞-property but not the S_∞-property. For the groups with infinite outer automorphism group, all groups without the S_∞-property also did not have the R_∞-property. The same quotient groups and automorphisms as in tables B.7 to B.11 can be used to obtain this result. We summarise these results in table A.1.

dim	# groups	no S_∞
1	2	1
2	17	2
3	219	13
4	4783	91

Table A.1: Crystallographic groups up to dimension 4 without the S_∞-property

A.2.2 Almost-crystallographic groups

In this section we determine which non-crystallographic almost-crystallographic groups of dimension 3 do not have the S_∞-property. We will use the same presentations as in section 9.2.1.

If Γ is a 3-dimensional non-crystallographic almost-crystallographic group, then the centre $Z(\Gamma)$ either equals the isolator $\sqrt[N]{\gamma_2(N)} = \langle e_1 \rangle$, or it is trivial. In the former case, we only need to consider those families of almost-crystallographic groups for which the quotient $\Gamma/\langle e_1 \rangle$ is a crystallographic group that does not have the R_∞-property, due to proposition A.1.6(4). In the latter case, we only need to consider those families whose groups do not have the R_∞-property themselves, due to proposition A.1.6(5).

Thus, the only families we need to consider here are the same three families we studied in section 9.2.1.

Family min.2-1.1-0. We have already shown in section 5.2.1 that the groups in this family do not have the R_∞-property, hence they do not have the S_∞-property either.

Family min.5-1.1-0. In this family, whether or not a group has the S_∞-property depends on the parameters k_1, k_2 and k_3.

Theorem A.2.4. *Let Γ be an almost-crystallographic group in family min.5-1.1-0. Then Γ has the S_∞-property if and only if $k_1 \equiv 0 \bmod 6$ and $k_2 \not\equiv k_3 \bmod 3$.*

Proof. We have determined in section 9.2.1 that an automorphism $\varphi : \Gamma \to \Gamma$ must be of the form

$$\begin{aligned}
\varphi(e_1) &= e_1^{\det(M)},\\
\varphi(e_2) &= e_1^{n_1} e_2^{m_1} e_3^{m_2},\\
\varphi(e_3) &= e_1^{n_2} e_2^{m_3} e_3^{m_4},\\
\varphi(\alpha) &= e_1^{n_3} e_2^{d_1} e_3^{d_2} \alpha^{\epsilon},
\end{aligned}$$

where

$$M = \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \quad n_i, d_j \in \mathbb{Z}, \quad \epsilon \in \{-1, 1\},$$

and note that the n_i will depend on M, d_1, d_2 and the parameters k_1, k_2, k_3. Let $\Gamma' := \Gamma / \langle e_1 \rangle$, i.e. the crystallographic group min.5-1.1-0, and let F' be its holonomy group. Then φ induces an automorphism $\varphi' = \xi_{(d,M)}$ on Γ'. If $\Phi \in \mathrm{Out}(\Gamma)$ contains φ, then by lemma A.1.3 and theorem 4.2.5 we have that

$$S(\Phi) < \infty \iff R(\varphi') < \infty \iff \det(\mathbb{1}_2 - A'M) \neq 0 \quad \forall A' \in F'.$$

Since $N_{\Gamma'}$ is finite, we can easily calculate those $M \in N_{\Gamma'}$ that satisfy this condition: they are given by

$$\{-A' \mid A' \in F'\} = \left\{ -\mathbb{1}_2, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Using the same techniques we used to prove proposition 10.2.1, we determine that an automorphism φ such that $M = -\mathbb{1}_2$ must be of the form

$$\begin{aligned}
\varphi(e_1) &= e_1,\\
\varphi(e_2) &= e_1^{\frac{1}{3}(-k_1(d_1+d_2+1)+4k_2+2k_3)} e_2^{-1},\\
\varphi(e_3) &= e_1^{\frac{1}{3}(k_1(2d_1-d_2-1)-2k_2+2k_3)} e_3^{-1},\\
\varphi(\alpha) &= e_1^{\frac{1}{6}(k_1(d_1+d_2)(d_1+d_2+1)-2k_2(2d_1-d_2)-2k_3(d_1+d_2))} e_2^{d_1} e_3^{d_2}\alpha,
\end{aligned}$$

where all powers must be integers. Similarly, an automorphism with $M = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ must be of the form

$$\begin{aligned}
\varphi(e_1) &= e_1,\\
\varphi(e_2) &= e_1^{\frac{1}{3}(k_1(2d_1-d_2-1)+k_2+2k_3)} e_3^{-1},\\
\varphi(e_3) &= e_1^{\frac{1}{3}(-k_1(d_1-2d_2-2)-2k_2-k_3)} e_2 e_3,\\
\varphi(\alpha) &= e_1^{\frac{1}{6}(k_1(d_1+d_2)(d_1+d_2+1)-2k_2(2d_1-d_2)-2k_3(d_1+d_2))} e_2^{d_1} e_3^{d_2}\alpha,
\end{aligned}$$

and an automorphism with $M = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ must be of the form

$$\begin{aligned}
\varphi(e_1) &= e_1,\\
\varphi(e_2) &= e_1^{\frac{1}{3}(-k_1(d_1-2d_2-2)+k_2-k_3)} e_2 e_3,\\
\varphi(e_3) &= e_1^{\frac{1}{3}(-k_1(d_1+d_2+1)+k_2+2k_3)} e_2^{-1},\\
\varphi(\alpha) &= e_1^{\frac{1}{6}(k_1(d_1+d_2)(d_1+d_2+1)-2k_2(2d_1-d_2)-2k_3(d_1+d_2))} e_2^{d_1} e_3^{d_2}\alpha.
\end{aligned}$$

Let us now take $M = -\mathbb{1}_2$ and fix an almost-crystallographic group with parameters k_1, k_2, k_3, k_4. If an automorphism φ induces $\varphi' = \xi_{(d,-\mathbb{1}_2)}$ on Γ', then d_1 and d_2 must satisfy the following conditions:

(a) $-k_1(d_1 + d_2 + 1) + 4k_2 + 2k_3 \equiv 0 \bmod 3$,

(b) $k_1(2d_1 - d_2 - 1) - 2k_2 + 2k_3 \equiv 0 \bmod 3$,

(c) $k_1(d_1 + d_2)(d_1 + d_2 + 1) - 2k_2(2d_1 - d_2) - 2k_3(d_1 + d_2) \equiv 0 \bmod 6$.

If such pair d_1, d_2 exists, then Γ does not have the S_∞-property. We use algorithm 12 to verify the existence of such d_1 and d_2. The result of this algorithm is that Γ admits an automorphism with $M = -\mathbb{1}_2$ if and only if either $k_1 \equiv 0 \bmod 6$ and $k_2 - k_3 \equiv 0 \bmod 3$, or $k_1 \not\equiv 0 \bmod 6$. Repeating the above steps for the other two choices of M, we obtain the exact same result. □

Algorithm 12 Determining whether an almost-crystallographic group in family min.5-1.1-0 admits an automorphism with $M = -\mathbb{1}_2$

function AdmitsAutomorphism(k_1, k_2, k_3, k_4)
    for $(d_1, d_2) \in \mathbb{Z}_6^2$ **do**
        if conditions (a), (b), (c) are met **then**
            return `true`
        end if
    end for
    return `false`
end function

Family group.1-1.1-0. We have already shown in section 9.2.1 that the groups in this family do not have the R_∞-property, hence they do not have the S_∞-property either.

A.3 The isogredience spectrum

In this section, we will determine the isogredience spectrum of the almost-crystallographic groups up to dimension 3.

A.3.1 Crystallographic groups

The crystallographic groups up to dimension 3 that do not have the S_∞-property were determined in the previous section. We calculate their isogredience spectrum below.

min.1-1.1-0, min.2-1.1-0, min.6-1.1-0. These are the groups $\Gamma = \mathbb{Z}^n$ with $n \in \{1, 2, 3\}$. For each of these groups we find that $\Gamma / Z(\Gamma)$ is trivial, so clearly $S(\Phi) = 1$ for any $\Phi \in \mathrm{Out}(\Gamma)$, and thus these groups have isogredience spectrum $\{1\}$.

min.5-1.1-0. This group has trivial centre, hence its isogredience spectrum equals its Reidemeister spectrum, which we already determined in section 10.1.1 to be $\{4, \infty\}$.

min.7-1.1-0, min.7-1.1-1. These are the groups $\Lambda = \Lambda_{3/2/\epsilon}$ with $\epsilon \in \{0, 1\}$ as described in chapter 7. For any $M \in \mathrm{GL}_2(\mathbb{Z})$ and $d' \in \mathbb{Z}^2$, we can construct the matrix $D \in \mathrm{GL}_3(\mathbb{Z})$ and vector $d \in (\frac{1}{2}\mathbb{Z})^3$ as

$$D = \begin{pmatrix} M & 0 \\ 0 & -1 \end{pmatrix}, d = \begin{pmatrix} d'/2 \\ 0 \end{pmatrix}$$

such that $\varphi = \xi_{(d,D)} \in \mathrm{Aut}(\Lambda)$. Then the automorphism induced by φ on $\Lambda/Z(\Lambda)$, i.e. the crystallographic group group.1-1.1-0, is $\varphi' = \xi_{(d'/2,M)}$. Because the projection $\mathrm{Aut}(\Lambda) \to \mathrm{Aut}(\Lambda/Z(\Lambda))$ is surjective, we have that

$$\mathrm{Spec}_S(\Lambda) = \mathrm{Spec}_R(\Lambda/Z(\Lambda)) = 2\mathbb{N} \cup \{3, \infty\}.$$

min.7-1.2-0. We use the presentation from section 10.1.2.

Theorem A.3.1. *The group min.7-1.2-0 has isogredience spectrum* $2\mathbb{N} \cup \{\infty\}$.

Proof. This group Γ has non-trivial centre, and the quotient $\Gamma' := \Gamma/Z(\Gamma)$ is the crystallographic group group.1-1.1-0. Let $\varphi \in \mathrm{Aut}(\Gamma)$ and $\Phi \in \mathrm{Out}(\Gamma)$ such that $\varphi \in \Phi$. As determined in proposition 10.1.6, φ induces an automorphism $\varphi' = \xi_{(d',D')}$ on Γ', where

$$D' = \begin{pmatrix} \varepsilon + 2m_1 & 2m_3 \\ m_2 & 1 + 2m_4 \end{pmatrix},$$

with $\varepsilon \in \{-1, 1\}$ and $m_i \in \mathbb{Z}$. In particular, the trace of D' is always even, hence by a reasoning similar to that in the proof of theorem 7.1.3 we find that $S(\Phi) = R(\varphi') \in 2\mathbb{N} \cup \infty$. Now, take the family of automorphisms φ_m on Γ we gave in the proof of theorem 10.1.7, take $\Phi_m \in \mathrm{Out}(\Gamma)$ such that $\varphi_m \in \Phi_m$ and let φ'_m be the induced automorphisms on Γ'. One can then determine, like in the proof of theorem 7.1.3, that

$$S(\Phi_m) = R(\varphi'_m) = 2m,$$

hence $\mathrm{Spec}_S(\Gamma) = 2\mathbb{N} \cup \{\infty\}$. ∎

min.10-1.1-0, min.10-1.1-3, min.10-1.3-0, min.10-1.4-0, min.10-1.4-1. As before, these groups have trivial centre, hence their isogredience spectrum equals their Reidemeister spectrum, which we already determined in section 10.1.1 to be $\{2, \infty\}$.

min.13-1.1-0, min.13-1.1-1, min.13-1.2-0. For each of these groups, $\Gamma/Z(\Gamma)$ is the group min.5-1.1-0, which has Reidemeister spectrum $\{4, \infty\}$. Since these groups do not have the S_∞-property and $\mathrm{Spec}_S(\Gamma) \subseteq \mathrm{Spec}_R(\Gamma/Z(\Gamma))$ by proposition A.1.6(2), their isogredience spectrum must be $\{4, \infty\}$.

group.1-1.1-0, group.5-1.1-0. These are the groups $\Gamma = \langle \mathbb{Z}^n, -\mathbb{1}_n \rangle$ with $n \in \{2, 3\}$. For each of these groups we find that $Z(\Gamma)$ is trivial, hence their isogredience spectra equal their Reidemeister spectra. We determined in theorem 7.1.3 that these spectra are $2\mathbb{N} \cup \{3, \infty\}$ and $\mathbb{N} \setminus \{1\} \cup \{\infty\}$ respectively.

We summarise these results in table A.2 below. Unlike in the crystallographic case, we do not omit the value $\{\infty\}$ from the spectra, since it is no longer the case that ∞ always belongs to the isogredience spectrum. We have indicated Bieberbach groups with a star (*).

CARAT	BBNWZ	IT	$\mathrm{Spec}_S(\Gamma)$
min.1-1.1-0*	1/1/1/1/1	1/1	$\{1\}$
min.2-1.1-0*	2/1/1/1/1	2/1	$\{1\}$
min.5-1.1-0	2/4/1/1/1	2/13	$\{4, \infty\}$
min.6-1.1-0*	3/1/1/1/1	3/1	$\{1\}$
min.7-1.1-0	3/2/1/1/1	3/3	$2\mathbb{N} \cup \{3, \infty\}$
min.7-1.1-1*	3/2/1/1/2	3/4	$2\mathbb{N} \cup \{3, \infty\}$
min.7-1.2-0	3/2/1/2/1	3/5	$2\mathbb{N} \cup \{\infty\}$
min.10-1.1-0	3/3/1/1/1	3/16	$\{2, \infty\}$
min.10-1.1-3*	3/3/1/1/2	3/19	$\{2, \infty\}$
min.10-1.3-0	3/3/1/3/1	3/22	$\{2, \infty\}$
min.10-1.4-0	3/3/1/4/1	3/23	$\{2, \infty\}$
min.10-1.4-1	3/3/1/4/2	3/24	$\{2, \infty\}$
min.13-1.1-0	3/5/1/2/1	3/143	$\{4, \infty\}$
min.13-1.1-1*	3/5/1/2/2	3/144	$\{4, \infty\}$
min.13-1.2-0	3/5/1/1/1	3/146	$\{4, \infty\}$
group.1-1.1-0	2/1/2/1/1	2/2	$2\mathbb{N} \cup \{3, \infty\}$
group.5-1.1-0	3/1/2/1/1	3/2	$\mathbb{N} \setminus \{1\} \cup \{\infty\}$

Table A.2: Crystallographic groups of dimension at most 3 that do not have the S_∞-property

A.3.2 Almost-crystallographic groups

We will use the same presentations as in section 9.2.1.

Family min.2-1.1-0.

Theorem A.3.2. *Let* Γ *be a* 3*-dimensional almost-crystallographic group in the family min.2-1.1-0. Then* Γ *has full isogredience spectrum.*

Proof. For any $m \in \mathbb{N}$, consider the automorphism φ_m given by

$$\varphi_m(e_1) = e_1^{-1}, \quad \varphi_m(e_2) = e_3, \quad \varphi_m(e_3) = e_2 e_3^{-m},$$

and let $\Phi_m \in \mathrm{Out}(\Gamma)$ be such that $\varphi_m \in \Phi_m$. Then the matrix D_m, given by

$$D_m = \begin{pmatrix} 0 & 1 \\ 1 & -m \end{pmatrix},$$

is the induced automorphism on $\Gamma/Z(\Gamma) \cong \mathbb{Z}^2$. Then $S(\Phi_m) = R(D_m) = |\det(\mathbb{1}_2 - D_m)| = m$, and thus $\mathrm{Spec}_S(\Gamma) = \mathbb{N} \cup \{\infty\}$. ☐

Family min.5-1.1-0. In the previous subsection, we determined which groups Γ in this family do not have the S_∞-property. Since the Reidemeister spectrum of $\Gamma/Z(\Gamma)$, i.e. the crystallographic group min.5-1.1-0, is $\{4, \infty\}$, then by lemma A.1.3 we have that $\mathrm{Spec}_S(\Gamma) = \{4, \infty\}$ if Γ does not have the S_∞-property.

Family group.1-1.1-0.

Theorem A.3.3. *Let* Γ *be a* 3*-dimensional almost-crystallographic group in the family group.1-1.1-0, with parameters* k_1, k_2, k_3, k_4*. Then the isogredience spectrum of* Γ *is*

- $2\mathbb{N} + 2 \cup \{\infty\}$*, if* $\bar{k}_1 = 0$ *and* $(\bar{k}_2, \bar{k}_3) \neq (0, 0)$*,*
- $2\mathbb{N} \cup \{3, \infty\}$*, if* $\bar{k}_1 = 1$ *or* $(\bar{k}_2, \bar{k}_3) = (0, 0)$*,*

where the bar-notation stands for the projection to $\mathbb{Z}_2$*.*

Proof. Let $\varphi : \Gamma \to \Gamma$ be an automorphism. Similar to what we did in the proof of proposition 10.2.1, we can calculate what φ must be like. If it maps e_1 to its inverse, then

$$\varphi(e_1) = e_1^{-1},$$

$$\varphi(e_2) = e_1^{\frac{k_1}{2}(m_1m_2+m_1d_2-m_2d_1)-\frac{k_2}{2}(m_1+1)-\frac{k_3}{2}m_2} e_2^{m_1} e_3^{m_2},$$

$$\varphi(e_3) = e_1^{\frac{k_1}{2}(m_3m_4+m_3d_2-m_4d_1)-\frac{k_2}{2}m_3-\frac{k_3}{2}(m_4+1)} e_2^{m_3} e_3^{m_4},$$

$$\varphi(\alpha) = e_1^{\frac{k_1}{2}d_1d_2-\frac{k_2}{2}d_1-\frac{k_3}{2}d_2-k_4} e_2^{d_1} e_3^{d_2}\alpha,$$

where all exponents must be integers. This places four conditions on the m_i and d_j:

(a) $k_1(m_1m_2 + m_1d_2 - m_2d_1) - k_2(m_1 + 1) - k_3m_2 \equiv 0 \mod 2$,

(b) $k_1(m_3m_4 + m_3d_2 - m_4d_1) - k_2m_3 - k_3(m_4 + 1) \equiv 0 \mod 2$,

(c) $k_1d_1d_2 - k_2d_1 - k_3d_2 \equiv 0 \mod 2$,

(d) $m_1m_4 - m_2m_3 = -1$.

If it maps e_1 to itself, it must be of the form

$$\varphi(e_1) = e_1,$$

$$\varphi(e_2) = e_1^{\frac{k_1}{2}(m_1m_2+m_1d_2-m_2d_1)-\frac{k_2}{2}(m_1-1)-\frac{k_3}{2}m_2} e_2^{m_1} e_3^{m_2},$$

$$\varphi(e_3) = e_1^{\frac{k_1}{2}(m_3m_4+m_3d_2-m_4d_1)-\frac{k_2}{2}m_3-\frac{k_3}{2}(m_4-1)} e_2^{m_3} e_3^{m_4},$$

$$\varphi(\alpha) = e_1^{\frac{k_1}{2}d_1d_2-\frac{k_2}{2}d_1-\frac{k_3}{2}d_2} e_2^{d_1} e_3^{d_2}\alpha,$$

where all exponents must be integers. This too places four conditions on the m_i and d_j:

(a) $k_1(m_1m_2 + m_1d_2 - m_2d_1) - k_2(m_1 - 1) - k_3m_2 \equiv 0 \mod 2$,

(b) $k_1(m_3m_4 + m_3d_2 - m_4d_1) - k_2m_3 - k_3(m_4 - 1) \equiv 0 \mod 2$,

(c) $k_1d_1d_2 - k_2d_1 - k_3d_2 \equiv 0 \mod 2$,

(d) $m_1m_4 - m_2m_3 = 1$.

We set
$$M = \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$
then the automorphism on the quotient $\Gamma/Z(\Gamma)$ induced by φ is $\varphi' = \xi_{(d/2,M)}$. Using lemma A.1.3, theorem 7.1.3 and taking the parity of $\operatorname{tr}(M)$ and the value of $\det(M)$ into account, we can further determine the possible isogredience numbers:

1. $\det(M) = -1$. Then the formula becomes
$$S(\Phi) = |\operatorname{tr}(M)|_\infty + O(\mathbb{1}_2 - M, d).$$
Depending on the value of $|\operatorname{tr}(M)|$, we have:
   (a) $|\operatorname{tr}(M)| \equiv 0 \bmod 2$, then $S(\Phi) \in 2\mathbb{N} + O(\mathbb{1}_2 - M, d)$,
   (b) $|\operatorname{tr}(M)| \equiv 1 \bmod 2$, then $S(\Phi) \in 2\mathbb{N}$.
2. $\det(M) = 1$. Then the formula becomes
$$S(\Phi) = \frac{|2 - \operatorname{tr}(M)|_\infty + |2 + \operatorname{tr}(M)|_\infty}{2} + O(\mathbb{1}_2 - M, 2d).$$
Depending on the value of $|\operatorname{tr}(M)|$, we have:
   (a) $|\operatorname{tr}(M)| \equiv 0 \bmod 2$, then $S(\Phi) \in 2\mathbb{N} + O(\mathbb{1}_2 - M, d)$,
   (b) $|\operatorname{tr}(M)| \equiv 1 \bmod 2$, then $S(\Phi) \in 2\mathbb{N} + 2 \cup \{3\}$.

There is one special case, however. If $M \equiv \mathbb{1}_2 \bmod 2$ all entries of $\mathbb{1}_2 - M$ will be multiples of 2; so $|\det(\mathbb{1}_2 - M)| = |\operatorname{tr}(M)| \in 4\mathbb{N}$ and therefore $S(\Phi) \in 4\mathbb{N} + O(\mathbb{1}_2 - M, d)$.

We will determine the isogredience spectrum in function of the parameters, similar to how we determined the Reidemeister spectrum in theorem 10.2.2. To this end, we will be using the function MakeList2 defined in algorithm 13, which is an isogredience analogue of the function MakeList defined in algorithm 10. The results can be found in tables B.13 to B.24. The isogredience spectrum of a group is a subset of (or the entirety of) the union of all these sets S.

Next, for each quadruplet of parameters, we found a family of automorphisms whose isogredience numbers produce the union of these sets S. These automorphisms and their isogredience numbers, for all (k_1, k_2, k_3, k_4), can be found in table A.3. For the sake of brevity, we omit ∞ from the spectra in this table. Note that all almost-Bieberbach groups belonging to this family have parameters with parities $(0, 0, 0, 1)$ and therefore have isogredience spectrum $2\mathbb{N} \cup \{3, \infty\}$. □

Algorithm 13 Determining automorphisms and isogredience spectra of 3-dimensional almost-crystallographic groups in family group.1-1.1-0

```
function MakeList2(k_1, k_2, k_3, k_4)
    AutList ← ∅
    for M̄ ∈ GL_2(Z_2), d̄ ∈ Z_2^2 do
        if conditions (a), (b), (c) are met then
            O ← 0
            for z̄ ∈ Z_2^2 do
                if M̄ z̄ = d̄ then
                    O ← O + 1
                end if
            end for
            for det ∈ {−1, 1} do
                if tr(M) ≡ 0 mod 2 then
                    if M ≡ 𝟙_2 mod 2 then
                        S ← 4N + O
                    else
                        S ← 2N + O
                    end if
                else
                    if det = −1 then
                        S ← 2N
                    else
                        S ← 2N + 2 ∪ {3}
                    end if
                end if
                AutList ← AutList ∪ {(M̄, d̄, det, S)}
            end for
        end if
    end for
    return AutList
end function
```

$(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4)$	M	d	$S(\Phi)$	$\mathrm{Spec}_S(\Gamma)$
$(0,0,0,0)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$2m$	$2\mathbb{N} \cup \{3\}$
	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	3	
$(0,0,0,1)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$2m$	$2\mathbb{N} \cup \{3\}$
	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	3	
$(0,0,1,0)$	$\begin{pmatrix} 1 & 1 \\ 2m & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$2m+2$	$2\mathbb{N}+2$
$(0,0,1,1)$	$\begin{pmatrix} 1 & 1 \\ 2m & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$2m+2$	$2\mathbb{N}+2$
$(0,1,1,0)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$2m+2$	$2\mathbb{N}+2$
$(0,1,1,1)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$2m+2$	$2\mathbb{N}+2$
$(1,0,0,0)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$2m$	$2\mathbb{N} \cup \{3\}$
	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	3	
$(1,0,0,1)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$2m$	$2\mathbb{N} \cup \{3\}$
	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	3	
$(1,0,1,0)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$2m$	$2\mathbb{N} \cup \{3\}$
	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	3	
$(1,0,1,1)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$2m$	$2\mathbb{N} \cup \{3\}$
	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	3	
$(1,1,1,0)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$2m$	$2\mathbb{N} \cup \{3\}$
	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	3	
$(1,1,1,0)$	$\begin{pmatrix} 0 & 1 \\ 1 & 2m-1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$2m$	$2\mathbb{N} \cup \{3\}$
	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	3	

Table A.3: Automorphisms and isogredience spectra for all (k_1, k_2, k_3, k_4) (we omit ∞ from the spectra)

Appendix B

Tables

This chapter only contains tables, usually presenting the output of some algorithm. Since many of these tables are long enough to span multiple pages, they were put here as an appendix to improve the readability of this thesis.

B.1 Crystallographic groups with finite outer automorphism group

The tables below contain the Reidemeister spectra of the crystallographic groups of dimensions 1 to 6 with finite outer automorphism group. These results were obtained using algorithms 3 and 9.

CARAT	BBNWZ	IT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
min.1-1.1-0*	1/1/1/1/1	1/1	2	$\{2\}$

Table B.1: Reidemeister spectra of the 1-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	BBNWZ	IT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
min.5-1.1-0	2/4/1/1/1	2/13	12	$\{4\}$

Table B.2: Reidemeister spectra of the 2-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	BBNWZ	IT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
min.10-1.1-0	3/3/1/1/1	3/16	96	$\{2\}$
min.10-1.1-3*	3/3/1/1/2	3/19	96	$\{2\}$
min.10-1.3-0	3/3/1/3/1	3/22	48	$\{2\}$
min.10-1.4-0	3/3/1/4/1	3/23	48	$\{2\}$
min.10-1.4-1	3/3/1/4/2	3/24	48	$\{2\}$
min.13-1.1-0	3/5/1/2/1	3/143	24	$\{8\}$
min.13-1.2-0	3/5/1/1/1	3/146	4	$\{8\}$

Table B.3: Reidemeister spectra of the 3-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	BBNWZ	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
min.28-1.1-0	4/22/7/2/1	8	$\{12\}$
min.32-1.1-0	4/22/1/2/1	288	$\{4, 16\}$
min.32-1.2-0	4/22/1/1/1	48	$\{16\}$
min.38-1.1-0	4/32/10/2/1	144	$\{6\}$
min.38-1.1-4	4/32/10/2/7	24	$\{6\}$
group.37-1.1-0	4/21/2/2/1	12	$\{3\}$
group.40-1.1-0	4/22/2/2/1	16	$\{8\}$
group.44-1.1-0	4/22/5/4/1	16	$\{6\}$
group.44-3.1-0	4/22/5/3/1	144	$\{6\}$
group.52-1.1-0	4/5/1/2/1	192	$\{4\}$
group.52-1.1-6*	4/5/1/2/9	192	$\{4\}$
group.52-1.3-0	4/5/1/9/1	96	$\{4\}$
group.52-1.6-0	4/5/1/13/1	48	$\{4\}$
group.52-1.7-0	4/5/1/5/1	96	$\{4\}$
group.52-1.7-1	4/5/1/5/2	96	$\{4\}$
group.52-1.12-0	4/5/1/7/1	96	$\{4\}$
group.52-1.12-3*	4/5/1/7/4	96	$\{4\}$
group.52-1.13-0	4/5/1/1/1	12	$\{4\}$
group.78-1.1-0	4/32/4/2/1	48	$\{2, 6\}$

Table B.4: Reidemeister spectra of the 4-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	BBNWZ	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.78-1.1-2	4/32/4/2/3	48	$\{2,6\}$
group.78-1.1-4	4/32/4/2/6	24	$\{2,6\}$
group.80-1.1-0	4/5/2/2/1	768	$\{2,4\}$
group.80-1.1-5	4/5/2/2/16	256	$\{2,4\}$
group.80-1.1-18	4/5/2/2/18	128	$\{2,4\}$
group.80-1.4-0	4/5/2/9/1	192	$\{4\}$
group.80-1.4-2	4/5/2/9/3	64	$\{4\}$
group.80-1.6-0	4/5/2/6/1	64	$\{4\}$
group.80-1.6-2	4/5/2/6/3	64	$\{4\}$
group.80-1.8-0	4/5/2/5/1	384	$\{4\}$
group.80-1.8-2	4/5/2/5/5	128	$\{4\}$
group.80-1.8-4	4/5/2/5/3	128	$\{2\}$
group.80-1.8-5	4/5/2/5/6	384	$\{2\}$
group.103-1.1-0	4/32/1/2/1	288	$\{2,6\}$
group.103-1.1-1	4/32/1/2/2	96	$\{2,6\}$
group.163-1.1-0	4/18/4/2/1	32	$\{4,8\}$
group.163-1.1-4	4/18/4/2/6	16	$\{4,8\}$
group.163-1.1-6	4/18/4/2/3	32	$\{4,8\}$
group.163-1.2-0	4/18/4/5/1	32	$\{4,8\}$
group.163-1.2-2	4/18/4/5/3	32	$\{4,8\}$
group.163-1.2-6	4/18/4/5/6	32	$\{4,8\}$
group.163-1.2-7	4/18/4/5/5	32	$\{4,8\}$
group.169-1.1-0	4/18/1/2/1	64	$\{4,8\}$
group.169-1.1-2	4/18/1/2/3	64	$\{4,8\}$
group.169-1.2-0	4/18/1/3/1	64	$\{4,8\}$
group.169-1.2-1	4/18/1/3/2	64	$\{4,8\}$

Table B.4: Reidemeister spectra of the 4-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
min.75-1.1-0	64	$\{8,16\}$
min.75-1.1-7	64	$\{8\}$
min.75-1.1-11	32	$\{8,16\}$
min.75-1.1-17	32	$\{8\}$
min.75-1.1-21	64	$\{8\}$
min.75-1.1-24	64	$\{8,16\}$
min.75-1.1-28*	32	$\{8\}$
min.75-1.1-31*	32	$\{8,16\}$

Table B.5: Reidemeister spectra of the 5-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
min.75-1.1-33	64	$\{8, 16\}$
min.75-1.1-35	64	$\{8\}$
min.75-1.1-36	64	$\{8\}$
min.75-1.1-37	64	$\{8, 16\}$
min.75-1.3-0	64	$\{8, 16\}$
min.75-1.3-2	64	$\{8\}$
min.75-1.3-14	64	$\{8\}$
min.75-1.3-15	64	$\{8, 16\}$
min.75-1.3-25	64	$\{8, 16\}$
min.75-1.3-27	64	$\{8\}$
min.75-1.3-31	64	$\{8\}$
min.75-1.3-32	64	$\{8, 16\}$
min.75-1.3-36	64	$\{8, 16\}$
min.75-1.3-38	64	$\{8\}$
min.75-1.3-43	64	$\{8\}$
min.75-1.3-44	64	$\{8, 16\}$
min.75-1.3-47	64	$\{8, 16\}$
min.75-1.3-49	64	$\{8\}$
min.75-1.3-50	64	$\{8\}$
min.75-1.3-51	64	$\{8, 16\}$
min.75-1.4-0	32	$\{8, 16\}$
min.75-1.4-1	32	$\{8, 16\}$
min.75-1.4-6	32	$\{8, 16\}$
min.75-1.4-7	32	$\{8, 16\}$
min.75-1.4-12	16	$\{4\}$
min.75-1.4-14*	16	$\{4\}$
min.75-1.5-0	32	$\{8, 16\}$
min.75-1.5-5	16	$\{8, 16\}$
min.75-1.5-7	32	$\{4\}$
min.75-1.5-10*	16	$\{4\}$
min.75-1.5-12	32	$\{8, 16\}$
min.75-1.5-13	32	$\{4\}$
min.119-1.1-0	1536	$\{4, 8\}$
min.119-1.1-3	512	$\{4, 8\}$
min.119-1.1-10	512	$\{4, 8\}$
min.119-1.1-11	512	$\{4, 8\}$
min.119-1.1-90	256	$\{4, 8\}$
min.119-1.1-113	256	$\{4, 8\}$
min.119-1.7-0	384	$\{8\}$
min.119-1.7-2	128	$\{8\}$

Table B.5: Reidemeister spectra of the 5-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
min.119-1.7-7	128	$\{8\}$
min.119-1.7-10	128	$\{8\}$
min.119-1.8-0	192	$\{8\}$
min.119-1.8-1	64	$\{8\}$
min.119-1.8-5	32	$\{4\}$
min.119-1.13-0	128	$\{8\}$
min.119-1.13-1	128	$\{8\}$
min.119-1.13-9	128	$\{8\}$
min.119-1.13-10	128	$\{8\}$
min.119-1.21-0	64	$\{8\}$
min.119-1.21-2	64	$\{4\}$
min.119-1.21-4	64	$\{8\}$
min.119-1.21-5	64	$\{4\}$
min.119-1.25-0	768	$\{8\}$
min.119-1.25-2	256	$\{8\}$
min.119-1.25-6	256	$\{4\}$
min.119-1.25-13	256	$\{8\}$
min.119-1.25-16*	256	$\{4\}$
min.119-1.25-22	768	$\{4\}$
min.119-1.25-24	256	$\{8\}$
min.119-1.25-25*	256	$\{4\}$
min.119-1.29-0	128	$\{8\}$
min.119-1.29-5	128	$\{8\}$
min.119-1.29-8	128	$\{4\}$
min.119-1.29-9*	128	$\{4\}$
min.119-1.30-0	768	$\{4, 8\}$
min.119-1.30-2	256	$\{4, 8\}$
min.119-1.30-17*	128	$\{4, 8\}$
group.255-1.1-0	128	$\{8, 16\}$
group.255-1.1-2	128	$\{8, 16\}$
group.255-1.1-3	64	$\{8, 16\}$
group.255-1.1-5	64	$\{8, 16\}$
group.255-1.1-10	128	$\{8\}$
group.255-1.1-12*	128	$\{8\}$
group.255-1.3-0	128	$\{8, 16\}$
group.255-1.3-1	128	$\{8, 16\}$
group.255-1.3-2	64	$\{8, 16\}$
group.255-1.3-3	64	$\{8, 16\}$
group.255-1.3-9	128	$\{8\}$
group.255-1.3-10	128	$\{8\}$

Table B.5: Reidemeister spectra of the 5-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.255-1.4-0	64	$\{8,16\}$
group.255-1.4-3*	32	$\{4\}$
group.255-1.4-4	64	$\{8,16\}$
group.255-1.5-0	64	$\{8,16\}$
group.255-1.5-2	64	$\{4\}$
group.255-1.5-4	64	$\{8,16\}$
group.255-1.5-5*	64	$\{4\}$
group.316-1.1-0	1152	$\{8\}$
group.316-1.1-3	1152	$\{8\}$
group.316-1.3-0	576	$\{8\}$
group.316-1.4-0	576	$\{8\}$
group.316-1.4-1	576	$\{8\}$
group.355-1.1-0	7680	$\{2\}$
group.355-1.1-331	640	$\{2\}$
group.355-1.1-356*	320	$\{2\}$
group.355-1.1-359	1280	$\{2\}$
group.355-1.5-0	960	$\{2\}$
group.355-1.5-3	160	$\{2\}$
group.355-1.16-0	3840	$\{2\}$
group.355-1.16-25	320	$\{2\}$
group.355-1.16-33	640	$\{2\}$
group.355-1.16-34	160	$\{2\}$
group.461-1.1-0	24	$\{6\}$
group.461-1.1-3*	12	$\{6\}$
group.461-1.1-4	12	$\{6\}$
group.461-1.1-5*	24	$\{6\}$
group.485-1.1-0	32	$\{12\}$
group.485-1.1-5	32	$\{8\}$
group.485-3.1-0	288	$\{12\}$
group.485-3.1-1	288	$\{8\}$
group.488-1.1-0	32	$\{16\}$
group.488-1.1-3*	8	$\{8\}$
group.488-1.1-4*	8	$\{16\}$
group.488-1.1-5	32	$\{8\}$
group.528-1.1-0	16	$\{24\}$
group.528-1.1-1	16	$\{16\}$
group.528-1.1-2	16	$\{12\}$
group.528-1.1-3	16	$\{12\}$
group.587-1.1-0	576	$\{8,32\}$
group.587-1.1-2*	144	$\{8\}$

Table B.5: Reidemeister spectra of the 5-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.587-1.2-0	48	$\{32\}$
group.587-1.3-0	96	$\{32\}$
group.587-1.4-0	16	$\{32\}$
group.587-1.5-0	48	$\{32\}$
group.756-1.1-0	96	$\{4, 12\}$
group.756-1.1-10	96	$\{4, 12\}$
group.756-1.1-18	48	$\{4, 12\}$
group.756-1.1-26*	96	$\{4, 8\}$
group.756-1.1-29*	96	$\{4, 8\}$
group.756-1.1-31*	48	$\{4, 8\}$
group.794-1.1-0	288	$\{12\}$
group.794-1.1-15	48	$\{8\}$
group.794-1.1-39	48	$\{12\}$
group.794-1.1-41	48	$\{8\}$
group.794-1.1-46	24	$\{8\}$
group.861-2.1-0	576	$\{4, 12\}$
group.861-2.1-1	192	$\{4, 12\}$
group.861-2.1-2	192	$\{8\}$

Table B.5: Reidemeister spectra of the 5-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
min.331-1.1-0	288	$\{8, 9, 24\}$
min.331-1.1-1	24	$\{8\}$
min.331-1.1-3	48	$\{8\}$
min.331-2.1-0	72	$\{9, 24\}$
min.331-3.1-0	864	$\{8, 9, 24\}$
min.331-3.1-1	72	$\{8\}$
min.331-3.1-3	144	$\{8\}$
min.332-1.1-0	36	$\{10, 16\}$
min.332-2.1-0	36	$\{10, 16\}$
min.332-3.1-0	36	$\{10, 16\}$
group.1701-1.1-0	24	$\{16\}$
group.1701-1.1-1	12	$\{16\}$
group.1701-2.1-0	8	$\{16\}$
group.1701-2.1-1	4	$\{16\}$
group.1768-2.1-0	48	$\{16\}$
group.1768-2.1-1	4	$\{16\}$

Table B.6: Reidemeister spectra of the 6-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.1768-2.1-3	8	$\{16\}$
group.1768-3.1-0	144	$\{16\}$
group.1768-3.1-1	12	$\{16\}$
group.1768-3.1-3	24	$\{16\}$
group.1800-1.1-0	24	$\{32\}$
group.1800-2.1-0	24	$\{32\}$
group.1800-3.1-0	24	$\{32\}$
group.1807-1.1-0	4	$\{16\}$
group.1807-2.1-0	4	$\{16\}$
group.1807-3.1-0	28	$\{16\}$
group.2637-1.1-0	96	$\{4\}$
group.2648-1.1-0	96	$\{48\}$
group.2649-2.1-0	48	$\{4\}$
group.2655-2.1-0	96	$\{4\}$
group.2655-2.3-0	72	$\{4\}$
group.2655-2.4-0	216	$\{4\}$
group.2655-4.1-0	2592	$\{4\}$
group.2669-1.1-0	216	$\{3, 6, 9\}$
group.2669-1.3-0	24	$\{6\}$
group.2669-1.4-0	24	$\{6\}$
group.2684-1.1-0	144	$\{10, 16\}$
group.2684-1.1-1	72	$\{10, 16\}$
group.2689-1.1-0	144	$\{9\}$
group.2689-1.3-0	36	$\{9\}$
group.2689-1.4-0	36	$\{9\}$
group.2775-1.1-0	108	$\{9\}$
group.2778-1.1-0	10368	$\{4, 16, 64\}$
group.2778-1.1-6	864	$\{16\}$
group.2778-1.2-0	576	$\{64\}$
group.2778-1.3-0	288	$\{4, 64\}$
group.2778-1.4-0	864	$\{4, 64\}$
group.2781-1.1-0	144	$\{12\}$
group.2781-1.1-1	72	$\{12\}$
group.2793-1.1-0	72	$\{3, 6, 9\}$
group.2832-1.1-0	24	$\{4\}$
group.2854-1.1-0	192	$\{32\}$
group.2854-1.1-2	48	$\{32\}$
group.2859-1.1-0	1728	$\{24\}$
group.2859-3.1-0	192	$\{24\}$
group.2872-1.1-0	36	$\{9\}$

Table B.6: Reidemeister spectra of the 6-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.2907-1.1-0	7776	$\{3, 5, 6, 8, 9, 21, 24\}$
group.2907-1.1-1	1296	$\{5, 8\}$
group.2907-1.1-3	972	$\{3, 9\}$
group.2907-1.2-0	432	$\{21, 24\}$
group.2907-1.3-0	648	$\{3, 6, 9, 21, 24\}$
group.2907-1.3-2	324	$\{3\}$
group.2907-1.4-0	648	$\{3, 6, 9, 21, 24\}$
group.2907-1.4-1	324	$\{3\}$
group.2907-1.5-0	72	$\{6, 24\}$
group.3112-1.1-0	2304	$\{16\}$
group.3112-1.1-8	2304	$\{16\}$
group.3112-1.5-0	1152	$\{16\}$
group.3112-1.6-0	576	$\{16\}$
group.3112-1.7-0	384	$\{16\}$
group.3112-1.7-6	384	$\{16\}$
group.3112-1.12-0	1152	$\{16\}$
group.3112-1.12-1	1152	$\{16\}$
group.3112-1.16-0	192	$\{16\}$
group.3112-1.17-0	96	$\{16\}$
group.3112-1.18-0	1152	$\{16\}$
group.3112-1.18-7	1152	$\{16\}$
group.3112-1.22-0	192	$\{16\}$
group.3112-1.22-1	192	$\{16\}$
group.3112-1.24-0	144	$\{16\}$
group.3112-1.25-0	192	$\{16\}$
group.3112-1.25-3	192	$\{16\}$
group.3112-1.26-0	24	$\{16\}$
group.3128-1.1-0	9216	$\{8, 16\}$
group.3128-1.1-15	3072	$\{8, 16\}$
group.3128-1.1-18	1536	$\{8, 16\}$
group.3128-1.4-0	2304	$\{16\}$
group.3128-1.4-2	768	$\{16\}$
group.3128-1.6-0	768	$\{16\}$
group.3128-1.6-2	768	$\{16\}$
group.3128-1.8-0	4608	$\{16\}$
group.3128-1.8-2	1536	$\{8\}$
group.3128-1.8-4	1536	$\{16\}$
group.3128-1.8-5	4608	$\{8\}$
group.3618-1.1-0	192	$\{4\}$
group.3618-1.1-20	192	$\{4\}$

Table B.6: Reidemeister spectra of the 6-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.3618-1.1-96	96	$\{4\}$
group.3618-1.1-99	96	$\{4\}$
group.3618-1.1-102	192	$\{4\}$
group.3618-1.1-103	192	$\{4\}$
group.3618-1.3-0	96	$\{2,4\}$
group.3618-1.3-21	96	$\{2,4\}$
group.3618-1.4-0	96	$\{4\}$
group.3618-1.4-5	48	$\{4\}$
group.3618-1.4-6	96	$\{4\}$
group.3618-1.4-8	48	$\{4\}$
group.3618-1.4-10	96	$\{4\}$
group.3618-1.4-11	96	$\{4\}$
group.3624-1.1-0	384	$\{2,4\}$
group.3624-1.1-3	384	$\{2,4\}$
group.3624-1.3-0	192	$\{2,4\}$
group.3624-1.4-0	192	$\{2,4\}$
group.3624-1.4-1	192	$\{2,4\}$
group.3624-1.4-2	96	$\{4\}$
group.3624-1.5-0	48	$\{4\}$
group.3624-1.5-1	48	$\{4\}$
group.3640-1.1-0	768	$\{4\}$
group.3640-1.1-7	768	$\{4\}$
group.3640-1.1-9	192	$\{2\}$
group.3640-1.1-10	192	$\{2\}$
group.3640-1.3-0	384	$\{2,4\}$
group.3640-1.3-3	384	$\{2,4\}$
group.3640-1.4-0	384	$\{4\}$
group.3640-1.4-4	384	$\{4\}$
group.3657-2.1-0	384	$\{4\}$
group.3657-2.1-9	384	$\{4\}$
group.3657-2.1-10	48	$\{4\}$
group.3657-2.3-0	192	$\{2,4\}$
group.3657-2.3-3	24	$\{2,4\}$
group.3657-2.3-4	192	$\{2,4\}$
group.3657-2.4-0	192	$\{4\}$
group.3657-2.4-3	192	$\{4\}$
group.3657-2.4-4	96	$\{4\}$
group.3893-1.1-0	192	$\{2\}$
group.3893-1.1-5	192	$\{2\}$
group.3893-1.3-0	384	$\{2\}$

Table B.6: Reidemeister spectra of the 6-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.3893-4.1-0	48	$\{2\}$
group.3893-4.1-5	48	$\{2\}$
group.3893-4.2-0	96	$\{2\}$
group.3893-4.2-9	96	$\{2\}$
group.3893-6.1-0	768	$\{2\}$
group.3921-1.1-0	96	$\{2,4\}$
group.3921-1.1-31	48	$\{2,4\}$
group.3921-1.1-33	96	$\{2,4\}$
group.3921-1.3-0	96	$\{2,4\}$
group.3921-1.3-5	96	$\{2,4\}$
group.3921-4.1-0	24	$\{4\}$
group.3921-4.1-12	24	$\{4\}$
group.3921-4.2-0	48	$\{4\}$
group.3921-4.2-10	48	$\{2\}$
group.3921-4.2-16	48	$\{4\}$
group.3921-4.2-18	48	$\{2\}$
group.3921-6.1-0	96	$\{2,4\}$
group.3921-6.1-2	96	$\{2,4\}$
group.5162-1.1-0	3456	$\{8,24\}$
group.5162-1.1-1	1152	$\{8,24\}$
group.5186-1.1-0	768	$\{16,32\}$
group.5186-1.1-2	768	$\{16,32\}$
group.5186-1.2-0	768	$\{16,32\}$
group.5186-1.2-1	768	$\{16,32\}$
group.5320-1.1-0	576	$\{8,24\}$
group.5320-1.1-3	288	$\{8,24\}$
group.5320-1.1-5	576	$\{8,24\}$
group.5471-1.1-0	1728	$\{24\}$
group.5471-1.1-5	288	$\{24\}$
group.5557-1.1-0	384	$\{16,32\}$
group.5557-1.1-4	384	$\{16,32\}$
group.5557-1.1-5	192	$\{16,32\}$
group.5557-1.2-0	384	$\{16,32\}$
group.5557-1.2-2	384	$\{16,32\}$
group.5557-1.2-4	384	$\{16,32\}$
group.5557-1.2-5	384	$\{16,32\}$
group.6559-1.1-0	18432	$\{2,4\}$
group.6559-1.1-614	9216	$\{4\}$
group.6559-1.1-1158	1536	$\{4\}$
group.6559-1.1-1399	1536	$\{4\}$

Table B.6: Reidemeister spectra of the 6-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.6559-1.1-3436	1536	$\{4\}$
group.6559-1.1-3862	1536	$\{4\}$
group.6559-1.1-3882	1536	$\{2,4\}$
group.6559-1.1-3931	3072	$\{2,4\}$
group.6559-1.1-3975	768	$\{4\}$
group.6559-1.1-4089	1536	$\{4\}$
group.6559-1.1-4112	18432	$\{2,4\}$
group.6559-1.1-4170	1536	$\{2,4\}$
group.6559-1.1-4171	3072	$\{2,4\}$
group.6559-1.5-0	4608	$\{4\}$
group.6559-1.5-10	768	$\{4\}$
group.6559-1.5-34	768	$\{4\}$
group.6559-1.5-35	4608	$\{4\}$
group.6559-1.9-0	2304	$\{2,4\}$
group.6559-1.9-9	384	$\{2,4\}$
group.6559-1.11-0	4608	$\{4\}$
group.6559-1.11-43	4608	$\{4\}$
group.6559-1.11-133	768	$\{4\}$
group.6559-1.11-151	768	$\{4\}$
group.6559-1.11-357	768	$\{4\}$
group.6559-1.11-370	768	$\{4\}$
group.6559-1.11-397	4608	$\{4\}$
group.6559-1.11-406	4608	$\{4\}$
group.6559-1.11-408	768	$\{4\}$
group.6559-1.11-409	768	$\{4\}$
group.6559-1.11-473	768	$\{4\}$
group.6559-1.11-477	768	$\{4\}$
group.6559-1.11-571	768	$\{4\}$
group.6559-1.11-574	768	$\{4\}$
group.6559-1.11-590	768	$\{4\}$
group.6559-1.11-591	768	$\{4\}$
group.6559-1.11-598	384	$\{4\}$
group.6559-1.11-599	384	$\{4\}$
group.6559-1.11-614	384	$\{4\}$
group.6559-1.11-615	384	$\{4\}$
group.6559-1.35-0	4608	$\{2,4\}$
group.6559-1.38-0	192	$\{2,4\}$
group.6559-1.38-1	96	$\{4\}$
group.6559-1.38-3	192	$\{2,4\}$
group.6559-1.49-0	2304	$\{4\}$

Table B.6: Reidemeister spectra of the 6-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.6559-1.49-8	2304	$\{4\}$
group.6559-1.49-10	384	$\{4\}$
group.6559-1.49-12	384	$\{4\}$
group.6559-1.50-0	2304	$\{4\}$
group.6559-1.50-6	2304	$\{4\}$
group.6559-1.50-10	384	$\{4\}$
group.6559-1.50-13	384	$\{4\}$
group.6559-1.66-0	384	$\{2,4\}$
group.6559-1.66-7	192	$\{4\}$
group.6559-1.66-12	384	$\{2,4\}$
group.6559-1.71-0	192	$\{4\}$
group.6559-1.71-10	192	$\{4\}$
group.6559-1.71-16	192	$\{4\}$
group.6559-1.71-19	192	$\{4\}$
group.6559-1.81-0	4608	$\{2,4\}$
group.6559-1.81-1	2304	$\{4\}$
group.6559-1.81-6	384	$\{4\}$
group.6559-1.81-7	384	$\{4\}$
group.6559-1.81-8	384	$\{4\}$
group.6559-1.81-9	384	$\{4\}$
group.6559-1.81-10	4608	$\{2,4\}$
group.6559-1.81-29	192	$\{4\}$
group.6559-1.81-30	384	$\{2,4\}$
group.6559-1.81-33	768	$\{2,4\}$
group.6559-1.81-34	384	$\{4\}$
group.6559-1.81-36	384	$\{2,4\}$
group.6559-1.81-37	768	$\{2,4\}$
group.6559-1.84-0	768	$\{2,4\}$
group.6559-1.84-45	384	$\{4\}$
group.6559-1.84-95	768	$\{2,4\}$
group.6559-1.85-0	9216	$\{2,4\}$
group.6559-1.85-91	4608	$\{4\}$
group.6559-1.85-116	768	$\{4\}$
group.6559-1.85-152	768	$\{4\}$
group.6559-1.85-213	768	$\{2,4\}$
group.6559-1.85-221	384	$\{4\}$
group.6559-1.85-248	768	$\{4\}$
group.6559-1.85-257	1536	$\{2,4\}$
group.6559-1.85-285	768	$\{4\}$
group.6559-1.85-320	768	$\{2,4\}$

Table B.6: Reidemeister spectra of the 6-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.6559-1.85-321	1536	$\{2,4\}$
group.6559-1.85-322	768	$\{4\}$
group.6559-1.85-323	9216	$\{2,4\}$
group.6559-1.86-0	576	$\{4\}$
group.6559-1.86-2	96	$\{4\}$
group.6560-1.1-0	15360	$\{4\}$
group.6560-1.1-636	1280	$\{4\}$
group.6560-1.1-1281	2560	$\{4\}$
group.6560-1.1-3917	640	$\{4\}$
group.6560-1.5-0	1920	$\{4\}$
group.6560-1.5-15	320	$\{4\}$
group.6560-1.10-0	960	$\{4\}$
group.6560-1.10-11	160	$\{4\}$
group.6560-1.64-0	7680	$\{4\}$
group.6560-1.64-191	320	$\{4\}$
group.6560-1.64-301	640	$\{4\}$
group.6560-1.64-351	1280	$\{4\}$
group.6560-1.75-0	7680	$\{4\}$
group.6560-1.75-339	640	$\{4\}$
group.6560-1.75-340	1280	$\{4\}$
group.6560-1.75-359	320	$\{4\}$
group.6560-1.76-0	240	$\{4\}$
group.6560-1.76-3	40	$\{4\}$
group.6566-1.1-0	92160	$\{2,4\}$
group.6566-1.1-346	9216	$\{2,4\}$
group.6566-1.1-2784	3072	$\{2,4\}$
group.6566-1.1-4088	6144	$\{2,4\}$
group.6566-1.1-20822	2304	$\{2,4\}$
group.6566-1.1-21314	768	$\{2,4\}$
group.6566-1.1-24319	768	$\{2,4\}$
group.6566-1.1-25098	1536	$\{2,4\}$
group.6566-1.1-25110	768	$\{2,4\}$
group.6566-1.1-25111	768	$\{2,4\}$
group.6566-1.6-0	5760	$\{2,4\}$
group.6566-1.6-3	576	$\{2,4\}$
group.6566-1.6-12	384	$\{2,4\}$
group.6566-1.6-13	96	$\{2,4\}$
group.6566-1.15-0	576	$\{2,4\}$
group.6566-1.15-9	576	$\{2,4\}$
group.6566-1.17-0	384	$\{4\}$

Table B.6: Reidemeister spectra of the 6-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

CARAT	# Out(Γ)	$\mathrm{Spec}_R(\Gamma)$
group.6566-1.17-4	384	$\{4\}$
group.6566-1.24-0	768	$\{4\}$
group.6566-1.24-24	768	$\{4\}$
group.6566-1.24-34	192	$\{2\}$
group.6566-1.24-35	192	$\{2\}$
group.6566-1.33-0	2304	$\{4\}$
group.6566-1.33-146	384	$\{4\}$
group.6566-1.33-152	192	$\{4\}$
group.6566-1.33-154	384	$\{4\}$
group.6566-1.33-315	192	$\{4\}$
group.6566-1.33-323	2304	$\{4\}$
group.6566-1.35-0	1536	$\{2, 4\}$
group.6566-1.35-92	384	$\{2, 4\}$
group.6566-1.35-201	192	$\{2, 4\}$
group.6566-1.36-0	46080	$\{4\}$
group.6566-1.36-24	4608	$\{2\}$
group.6566-1.36-284	1536	$\{4\}$
group.6566-1.36-780	768	$\{2\}$
group.6566-1.36-815	3072	$\{4\}$
group.6566-1.36-860	384	$\{2\}$
group.6566-1.36-910	384	$\{4\}$
group.6566-1.36-922	384	$\{2\}$
group.6566-1.36-924	768	$\{4\}$
group.6566-1.36-925	2304	$\{2, 4\}$
group.6566-1.36-927	7680	$\{2\}$

Table B.6: Reidemeister spectra of the 6-dimensional crystallographic groups with finite outer automorphism group (we omit ∞ from the spectra, as well as the groups that have the R_∞-property)

B.2 Crystallographic groups with infinite outer automorphism group

The tables below contain ($\mathbb{Z}$-classes of) crystallographic groups with infinite outer automorphism group. The R_∞-property for these groups was studied in section 9.1.

B.2.1 Groups that do not have the R_∞-property

The tables below contain the crystallographic groups with infinite outer automorphism group of dimensions 2 to 4 that do not have the R_∞-property. For each group, we also list an automorphism with finite Reidemeister number.

CARAT	BBNWZ	IT	d	D
min.2-1.1-0*	2/1/1/1/1	1/1	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}-5&1\\1&0\end{pmatrix}$
group.1-1.1-0	2/1/2/1/1	2/2	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}-8&-5\\5&3\end{pmatrix}$

Table B.7: 2-dimensional crystallographic groups with infinite outer automorphism group that do not have the R_∞-property

CARAT	BBNWZ	IT	d	D
min.6-1.1-0*	3/1/1/1/1	3/1	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-3&0&1\\1&0&0\\0&1&0\end{pmatrix}$
min.7-1.1-0	3/2/1/1/1	3/3	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-5&-3&0\\2&1&0\\0&0&-1\end{pmatrix}$
min.7-1.1-1*	3/2/1/1/2	3/4	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-5&-3&0\\3&2&0\\0&0&-1\end{pmatrix}$
min.7-1.2-0	3/2/1/2/1	3/5	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-11&-7&-7\\4&2&3\\4&3&2\end{pmatrix}$
group.5-1.1-0	3/1/2/1/1	3/2	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-2&-1&0\\2&0&-1\\-1&0&1\end{pmatrix}$

Table B.8: 3-dimensional crystallographic groups with infinite outer automorphism group that do not have the R_∞-property

CARAT	BBNWZ	d	D
min.15-1.1-0*	4/1/1/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-3&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$
min.17-1.1-0	4/2/2/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-2&-1&-1&0\\1&0&0&0\\1&0&1&0\\0&0&0&-1\end{pmatrix}$
min.17-1.1-1*	4/2/2/1/2	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-2&-1&-1&0\\1&1&0&0\\1&0&0&0\\0&0&0&-1\end{pmatrix}$
min.17-1.2-0	4/2/2/2/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-1&0&-1&-1\\-1&1&0&0\\0&-1&-1&0\\0&-1&0&-1\end{pmatrix}$
min.18-1.1-0	4/3/1/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-2&-1&0&0\\1&0&0&0\\0&0&-2&-1\\0&0&1&1\end{pmatrix}$

Table B.9: 4-dimensional crystallographic groups with infinite outer automorphism group that do not have the R_∞-property

CARAT	BBNWZ	d	D
min.18-1.1-1*	4/3/1/1/2	$\begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
min.18-1.2-0	4/3/1/2/1	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$
min.18-1.2-1*	4/3/1/2/2	$\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$
min.18-1.3-0	4/3/1/3/1	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -2 & -2 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
min.36-1.1-0	4/10/1/1/1	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -3 & -2 & -3 & 0 \\ 3 & 3 & 2 & 1 \\ 1 & -2 & 3 & -3 \\ 0 & -3 & 2 & -3 \end{pmatrix}$
min.43-1.1-0	4/28/1/1/1	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -4 & -3 & -3 & 1 \\ 1 & 4 & -1 & -4 \\ 4 & 1 & 4 & 1 \\ -1 & 3 & -3 & -4 \end{pmatrix}$
min.44-1.1-0	4/28/2/1/1	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -8 & -4 & -3 & 4 \\ 4 & 8 & -1 & -3 \\ -3 & 1 & -2 & 1 \\ -4 & -3 & -1 & 2 \end{pmatrix}$
max.6-1.1-0	4/26/2/1/1	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -29 & -41 & 0 & -29 \\ -41 & -29 & 29 & 0 \\ 0 & 29 & 29 & 41 \\ -29 & 0 & 41 & 29 \end{pmatrix}$
max.6-1.1-1	4/26/2/1/2	$\begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} -29 & 41 & 0 & -29 \\ 41 & -29 & 29 & 0 \\ 0 & 29 & 29 & -41 \\ -29 & 0 & -41 & 29 \end{pmatrix}$
group.26-1.1-0	4/1/2/1/1	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -2 & -1 & -2 & -1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix}$
group.28-1.1-0	4/3/2/1/1	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -2 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$
group.28-1.1-1	4/3/2/1/2	$\begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -2 \end{pmatrix}$
group.28-1.1-2	4/3/2/1/3	$\begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$
group.28-1.2-0	4/3/2/2/1	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & -1 & -1 \\ 0 & -3 & -5 & 5 \\ 1 & 1 & 2 & -1 \\ 1 & -1 & -1 & 2 \end{pmatrix}$
group.28-1.2-1	4/3/2/2/2	$\begin{pmatrix} 0 \\ 0 \\ 3/4 \\ 3/4 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$
group.28-1.2-2	4/3/2/2/3	$\begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & -1 & -1 \\ 0 & -3 & -5 & 5 \\ 1 & 1 & 2 & -1 \\ 1 & -1 & -1 & 2 \end{pmatrix}$

Table B.9: 4-dimensional crystallographic groups with infinite outer automorphism group that do not have the R_∞-property

CARAT	BBNWZ	d	D
group.28-1.3-0	4/3/2/3/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-1&-2&-1&-1\\1&1&0&1\\1&1&1&2\\0&-1&-1&-1\end{pmatrix}$
group.96-1.1-0	4/16/1/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-5&-2&0&0\\2&1&0&0\\0&0&-1&-2\\0&0&2&5\end{pmatrix}$
group.96-1.1-1	4/16/1/1/2	$\begin{pmatrix}1/2\\0\\0\\1/2\end{pmatrix}$	$\begin{pmatrix}-3&-2&0&0\\1&1&0&0\\0&0&1&1\\0&0&-2&-3\end{pmatrix}$
group.96-2.1-0	4/16/1/2/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-41&0&-12&-12\\0&-41&-12&12\\12&12&7&0\\12&-12&0&7\end{pmatrix}$
group.96-2.1-1	4/16/1/2/2	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-7&0&-2&-2\\0&-7&-2&2\\2&2&1&0\\2&-2&0&1\end{pmatrix}$
group.96-2.1-2	4/16/1/2/3	$\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}$	$\begin{pmatrix}-41&0&-12&-12\\0&-41&-12&12\\12&12&7&0\\12&-12&0&7\end{pmatrix}$
group.96-3.1-0	4/16/1/3/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-4&-1&0&4\\-1&6&12&-2\\0&-2&-4&1\\-2&0&1&2\end{pmatrix}$
group.109-1.1-0	4/26/1/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-41&29&-29&0\\29&-41&0&-29\\-29&0&-41&-29\\0&-29&-29&-41\end{pmatrix}$
group.141-1.1-0	4/27/2/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-3&0&2&2\\1&1&-2&0\\3&-2&0&-3\\-1&2&-1&2\end{pmatrix}$
group.142-1.1-0	4/27/3/2/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-3&0&2&2\\1&1&-2&0\\3&-2&0&-3\\-1&2&-1&2\end{pmatrix}$
group.142-2.1-0	4/27/3/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-5&-5&-2&-3\\0&5&-3&0\\0&-3&2&0\\-3&-5&0&-2\end{pmatrix}$
group.143-1.1-0	4/27/4/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-3&0&2&2\\1&1&-2&0\\3&-2&0&-3\\-1&2&-1&2\end{pmatrix}$
group.144-1.1-0	4/27/1/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-3&0&2&2\\1&1&-2&0\\3&-2&0&-3\\-1&2&-1&2\end{pmatrix}$
group.170-1.1-0	4/11/1/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-2&0&-1&0\\-2&2&-1&1\\1&0&1&0\\1&-1&1&-1\end{pmatrix}$
group.171-1.1-0	4/11/2/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-2&-1&-1&-1\\1&-1&0&1\\0&1&0&-1\\1&0&1&1\end{pmatrix}$
group.172-1.1-0	4/17/2/2/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-3&-4&-4&-2\\4&1&2&-2\\4&2&3&-1\\-2&2&1&4\end{pmatrix}$

Table B.9: 4-dimensional crystallographic groups with infinite outer automorphism group that do not have the R_∞-property

CARAT	BBNWZ	d	D
group.172-2.1-0	4/17/2/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-3&-2&-1&0\\0&2&0&1\\2&2&1&0\\0&1&0&0\end{pmatrix}$
group.173-1.1-0	4/17/1/3/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-13&-2&-14&-8\\2&-11&8&-6\\14&6&13&11\\-6&8&-11&2\end{pmatrix}$
group.173-2.1-0	4/17/1/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-3&-2&-1&0\\3&0&1&-1\\3&1&1&-1\\2&0&1&-1\end{pmatrix}$
group.173-3.1-0	4/17/1/2/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-4&1&-4&5\\0&0&1&0\\0&-1&3&0\\-1&0&-1&1\end{pmatrix}$
group.179-1.1-0	4/8/1/2/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-2&-1\\0&0&1&1\end{pmatrix}$
group.179-1.1-1*	4/8/1/2/2	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&1\\0&0&-3&-2\end{pmatrix}$
group.179-1.2-0	4/8/1/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-5&-3&-3&3\\1&0&1&-1\\1&1&0&-1\\-1&-1&-1&0\end{pmatrix}$
group.179-1.2-1*	4/8/1/1/2	$\begin{pmatrix}1/3\\1/3\\2/3\\0\end{pmatrix}$	$\begin{pmatrix}-2&-3&-3&3\\1&1&2&-2\\1&2&1&-2\\-1&-2&-2&1\end{pmatrix}$
group.182-1.1-0	4/9/2/1/1	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-2&-1\\0&0&1&1\end{pmatrix}$

Table B.9: 4-dimensional crystallographic groups with infinite outer automorphism group that do not have the R_∞-property

B.2.2 Groups that have the R_∞-property

The tables below contain the $\mathbb{Z}$-classes of crystallographic groups with infinite outer automorphism group of dimensions 3 and 4 that have the R_∞-property. For each group $\mathbb{Z}^n \rtimes F$, we also list a characteristic subgroup N such that the quotient has the R_∞-property.

CARAT	BBNWZ	IT	N	$(\mathbb{Z}^3 \rtimes F)/N$
min.8-1.1	3/2/2/1	3/6, 3/7	$\mathbb{Z}^2$	max.1-1.1-0
min.8-1.2	3/2/2/2	3/8, 3/9	$\mathbb{Z}^2$	max.1-1.1-0
group.6-1.1	3/2/3/1	3/10, 3/11, 3/13, 3/14	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	max.1-1.1-0
group.6-1.2	3/2/3/2	3/12, 3/15	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	max.1-1.1-0

Table B.10: 3-dimensional $\mathbb{Z}$-classes of crystallographic groups with infinite outer automorphism group that have the R_∞-property

CARAT	BBNWZ	N	$(\mathbb{Z}^4 \rtimes F)/N$
min.16-1.1	4/2/1/1	$\mathbb{Z}^3$	max.1-1.1-0
min.16-1.2	4/2/1/2	$\mathbb{Z}^3$	max.1-1.1-0
min.19-1.1	4/4/3/1	$\mathbb{Z}^2$	group.2-1.1-0
min.19-1.2	4/4/3/3	$\mathbb{Z}^2$	group.2-1.1-0
min.19-1.3	4/4/3/2	$\mathbb{Z}^2$	group.2-1.2-0
min.19-1.4	4/4/3/6	$\mathbb{Z}^2$	group.2-1.1-0
min.19-1.5	4/4/3/4	$\mathbb{Z}^2$	group.2-1.2-0
min.19-1.6	4/4/3/5	$\mathbb{Z}^2$	group.2-1.1-0
min.20-1.1	4/4/1/1	$\mathbb{Z}^2$	group.2-1.1-0
min.20-1.2	4/4/1/3	$\mathbb{Z}^2$	group.2-1.1-0
min.20-1.3	4/4/1/2	$\mathbb{Z}^2$	group.2-1.2-0
min.20-1.4	4/4/1/6	$\mathbb{Z}^2$	group.2-1.2-0
min.20-1.5	4/4/1/4	$\mathbb{Z}^2$	group.2-1.1-0
min.20-1.6	4/4/1/5	$\mathbb{Z}^2$	group.2-1.1-0
min.21-1.1	4/4/2/1	$\mathbb{Z}$	group.6-1.1-0
min.21-1.2	4/4/2/4	$\mathbb{Z}$	group.6-1.1-0
min.21-1.3	4/4/2/3	$\mathbb{Z}$	group.6-1.2-0
min.21-1.4	4/4/2/2	$\mathbb{Z}$	group.6-1.1-0
min.21-1.5	4/4/2/7	$\mathbb{Z}$	group.6-1.1-0
min.21-1.6	4/4/2/5	$\mathbb{Z}$	group.6-1.2-0
min.21-1.7	4/4/2/6	$\mathbb{Z}$	group.6-1.2-0
min.26-1.1	4/4/4/1	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	group.2-1.1-0
min.26-1.2	4/4/4/3	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	group.2-1.1-0
min.26-1.3	4/4/4/2	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	group.2-1.2-0
min.26-1.4	4/4/4/6	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	group.2-1.1-0
min.26-1.5	4/4/4/4	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	group.2-1.2-0
min.26-1.6	4/4/4/5	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	group.2-1.1-0
min.42-1.1	4/7/1/1	$\mathbb{Z}^2$	min.4-1.1-0
min.42-1.2	4/7/1/2	$\mathbb{Z}^2$	min.4-1.1-0
min.48-1.1	4/9/4/1	$\mathbb{Z}^2$	max.3-1.1-0
min.49-1.1	4/8/3/2	$\mathbb{Z}^2$	group.4-2.1-0

Table B.11: 4-dimensional $\mathbb{Z}$-classes of crystallographic groups with infinite outer automorphism group that have the R_∞-property

CARAT	BBNWZ	N	$(\mathbb{Z}^4 \rtimes F)/N$
min.49-2.1	4/8/3/3	$\mathbb{Z}^2$	group.4-1.1-0
min.49-2.2	4/8/3/1	$\mathbb{Z}^2$	group.4-1.1-0
min.50-1.1	4/8/4/2	$\mathbb{Z}^2$	group.4-2.1-0
min.50-1.2	4/8/4/1	$\mathbb{Z}^2$	group.4-1.1-0
min.50-2.1	4/8/4/3	$\mathbb{Z}^2$	group.4-1.1-0
group.27-1.1	4/2/3/1	$\mathbb{Z}^3 \rtimes \mathbb{Z}_2$	max.1-1.1-0
group.27-1.2	4/2/3/2	$\mathbb{Z}^3 \rtimes \mathbb{Z}_2$	max.1-1.1-0
group.107-1.1	4/7/7/1	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	max.2-1.1-0
group.107-1.2	4/7/7/2	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	max.2-1.1-0
group.117-1.1	4/7/2/1	$\mathbb{Z}^2$	min.4-1.1-0
group.117-1.2	4/7/2/2	$\mathbb{Z}^2$	min.4-1.1-0
group.136-1.1	4/7/3/1	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	min.4-1.1-0
group.136-1.2	4/7/3/2	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	min.4-1.1-0
group.145-1.1	4/7/5/1	$\mathbb{Z}^2$	max.2-1.1-0
group.145-1.2	4/7/5/2	$\mathbb{Z}^2$	max.2-1.1-0
group.146-1.1	4/7/6/1	$\mathbb{Z}^2$	max.2-1.1-0
group.146-1.2	4/7/6/2	$\mathbb{Z}^2$	max.2-1.1-0
group.147-1.1	4/7/4/1	$\mathbb{Z}^2$	max.2-1.1-0
group.147-1.2	4/7/4/3	$\mathbb{Z}^2$	max.2-1.1-0
group.147-2.1	4/7/4/2	$\mathbb{Z}^2$	max.2-1.1-0
group.147-2.2	4/7/4/4	$\mathbb{Z}^2$	max.2-1.1-0
group.174-1.1	4/8/5/3	$\mathbb{Z}^2$	max.3-1.1-0
group.174-1.2	4/8/5/1	$\mathbb{Z}^2$	max.3-1.1-0
group.174-2.1	4/8/5/2	$\mathbb{Z}^2$	max.3-1.1-0
group.175-1.1	4/9/3/1	$\mathbb{Z}^2$	group.3-1.1-0
group.176-1.1	4/9/5/1	$\mathbb{Z}^2$	max.3-1.1-0
group.177-1.1	4/9/6/1	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	group.4-2.1-0
group.177-2.1	4/9/6/2	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	group.4-1.1-0
group.178-1.1	4/9/7/1	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	max.3-1.1-0
group.180-1.1	4/8/2/2	$\mathbb{Z}^2$	group.3-1.1-0
group.180-1.2	4/8/2/1	$\mathbb{Z}^2$	group.3-1.1-0
group.181-1.1	4/9/1/1	$\mathbb{Z}^2$	group.3-1.1-0

Table B.11: 4-dimensional $\mathbb{Z}$-classes of crystallographic groups with infinite outer automorphism group that have the R_∞-property

B.3 Almost-crystallographic groups

The tables below contain information on the almost-crystallographic groups of which we determined the R_∞-property and/or the Reidemeister spectrum.

B.3.1 Conjugacy matrices

The table below contains conjugacy matrices to go from the presentations mentioned in this thesis to those given in [Dek96] and [DE02].

Family	δ
min.6-1.1-0	$\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$
min.7-1.1-0	$\begin{pmatrix} 1&0&0&0&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$
min.7-1.1-1	$\begin{pmatrix} 1&0&0&0&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$
min.7-1.2-0	$\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&1&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$
min.13-1.1-0	$\begin{pmatrix} 1 & -\frac{k_1}{2}+k_2+2k_3 & -k_2+k_3 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
min.13-1.2-0	$\begin{pmatrix} 1&0&0&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1 \end{pmatrix}$
group.5-1.1-0	$\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$

Table B.12: Conjugacy matrices between representations of 4-dimensional almost-crystallographic groups

B.3.2 Automorphisms of family group.1-1.1-0

The tables below contain the output of algorithms 10 and 13.

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+4$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}+4$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$8\mathbb{N}+8$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	-1	$8\mathbb{N}$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	-1	$8\mathbb{N}$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$8\mathbb{N}$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+4$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}+4$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+4$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+4$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}$

Table B.13: Output of MakeList$(0,0,0,0)$ and MakeList2$(0,0,0,0)$

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\binom{1\ 1}{1\ 0}$	$\binom{0}{0}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\binom{1\ 1}{1\ 0}$	$\binom{0}{0}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\binom{1\ 1}{1\ 0}$	$\binom{0}{1}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\binom{1\ 1}{1\ 0}$	$\binom{0}{1}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\binom{1\ 1}{1\ 0}$	$\binom{1}{0}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\binom{1\ 1}{1\ 0}$	$\binom{1}{0}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\binom{1\ 1}{1\ 0}$	$\binom{1}{1}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\binom{1\ 1}{1\ 0}$	$\binom{1}{1}$	1	∞	$2\mathbb{N}+2\cup\{3\}$

Table B.13: Output of MakeList(0, 0, 0, 0) and MakeList2(0, 0, 0, 0)

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\binom{0\ 1}{1\ 0}$	$\binom{0}{0}$	-1	$4\mathbb{N}$	$2\mathbb{N}+2$
$\binom{0\ 1}{1\ 0}$	$\binom{0}{0}$	1	∞	$2\mathbb{N}+2$
$\binom{0\ 1}{1\ 0}$	$\binom{0}{1}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\binom{0\ 1}{1\ 0}$	$\binom{0}{1}$	1	∞	$2\mathbb{N}$
$\binom{0\ 1}{1\ 0}$	$\binom{1}{0}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\binom{0\ 1}{1\ 0}$	$\binom{1}{0}$	1	∞	$2\mathbb{N}$
$\binom{0\ 1}{1\ 0}$	$\binom{1}{1}$	-1	$4\mathbb{N}$	$2\mathbb{N}+2$
$\binom{0\ 1}{1\ 0}$	$\binom{1}{1}$	1	∞	$2\mathbb{N}+2$
$\binom{0\ 1}{1\ 1}$	$\binom{0}{0}$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\binom{0\ 1}{1\ 1}$	$\binom{0}{0}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\binom{0\ 1}{1\ 1}$	$\binom{0}{1}$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\binom{0\ 1}{1\ 1}$	$\binom{0}{1}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\binom{0\ 1}{1\ 1}$	$\binom{1}{0}$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\binom{0\ 1}{1\ 1}$	$\binom{1}{0}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\binom{0\ 1}{1\ 1}$	$\binom{1}{1}$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\binom{0\ 1}{1\ 1}$	$\binom{1}{1}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\binom{1\ 0}{0\ 1}$	$\binom{0}{0}$	-1	$8\mathbb{N}$	$4\mathbb{N}+4$
$\binom{1\ 0}{0\ 1}$	$\binom{0}{0}$	1	∞	$4\mathbb{N}+4$
$\binom{1\ 0}{0\ 1}$	$\binom{0}{1}$	-1	$8\mathbb{N}$	$4\mathbb{N}$
$\binom{1\ 0}{0\ 1}$	$\binom{0}{1}$	1	∞	$4\mathbb{N}$
$\binom{1\ 0}{0\ 1}$	$\binom{1}{0}$	-1	$8\mathbb{N}$	$4\mathbb{N}$
$\binom{1\ 0}{0\ 1}$	$\binom{1}{0}$	1	∞	$4\mathbb{N}$
$\binom{1\ 0}{0\ 1}$	$\binom{1}{1}$	-1	$8\mathbb{N}$	$4\mathbb{N}$
$\binom{1\ 0}{0\ 1}$	$\binom{1}{1}$	1	∞	$4\mathbb{N}$
$\binom{1\ 0}{1\ 1}$	$\binom{0}{0}$	-1	$4\mathbb{N}$	$2\mathbb{N}+2$
$\binom{1\ 0}{1\ 1}$	$\binom{0}{0}$	1	∞	$2\mathbb{N}+2$
$\binom{1\ 0}{1\ 1}$	$\binom{0}{1}$	-1	$4\mathbb{N}$	$2\mathbb{N}+2$
$\binom{1\ 0}{1\ 1}$	$\binom{0}{1}$	1	∞	$2\mathbb{N}+2$

Table B.14: Output of MakeList(0, 0, 0, 1) and MakeList2(0, 0, 0, 1)

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	1	∞	$2\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}$	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	1	∞	$2\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}$	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	1	∞	$2\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	1	∞	$2\mathbb{N}+2\cup\{3\}$

Table B.14: Output of MakeList$(0,0,0,1)$ and MakeList2$(0,0,0,1)$

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	-1	$8\mathbb{N}+4$	$4\mathbb{N}+4$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	1	∞	$4\mathbb{N}+4$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	-1	$8\mathbb{N}$	$4\mathbb{N}$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	1	∞	$4\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}+4$	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}$	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2$

Table B.15: Output of MakeList$(0,0,1,0)$ and MakeList2$(0,0,1,0)$

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$8\mathbb{N}+4$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	-1	$8\mathbb{N}$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+4$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$

Table B.16: Output of MAKELIST$(0,0,1,1)$ and MAKELIST2$(0,0,1,1)$

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+4$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$8\mathbb{N}+4$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$8\mathbb{N}$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}$

Table B.17: Output of MAKELIST$(0,1,1,0)$ and MAKELIST2$(0,1,1,0)$

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}+4$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$8\mathbb{N}+4$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$8\mathbb{N}$	$4\mathbb{N}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}$

Table B.18: Output of MAKELIST$(0,1,1,1)$ and MAKELIST2$(0,1,1,1)$

$\bar{M}$	$\bar{d}$	det(M)	R	S
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2$
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	-1	$8\mathbb{N}+6$	$4\mathbb{N}+4$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	1	∞	$4\mathbb{N}+4$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	1	∞	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2\cup\{3\}$

Table B.19: Output of MakeList$(1,0,0,0)$ and MakeList2$(1,0,0,0)$

$\bar{M}$	$\bar{d}$	det(M)	R	S
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2$
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	-1	$8\mathbb{N}+2$	$4\mathbb{N}+4$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	1	∞	$4\mathbb{N}+4$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	1	∞	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	1	∞	$2\mathbb{N}+2\cup\{3\}$

Table B.20: Output of MakeList$(1,0,0,1)$ and MakeList2$(1,0,0,1)$

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$8\mathbb{N}+6$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$

Table B.21: Output of MakeList$(1,0,1,0)$ and MakeList2$(1,0,1,0)$

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$8\mathbb{N}+2$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$

Table B.22: Output of MakeList$(1,0,1,1)$ and MakeList2$(1,0,1,1)$

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$8\mathbb{N}+2$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}$	$2\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$

Table B.23: Output of MakeList(1, 1, 1, 0) and MakeList2(1, 1, 1, 0)

$\bar{M}$	$\bar{d}$	$\det(M)$	R	S
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$8\mathbb{N}+6$	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$4\mathbb{N}+4$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}+2$	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	-1	$4\mathbb{N}-2$	$2\mathbb{N}$
$\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	1	∞	$2\mathbb{N}+2\cup\{3\}$

Table B.24: Output of MakeList(1, 1, 1, 1) and MakeList2(1, 1, 1, 1)

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