ALGORITHMS FOR TWISTED CONJUGACY CLASSES OF POLYCYCLIC-BY-FINITE GROUPS II

SAM TERTOOY

Abstract. We construct an algorithm that, given a pair of homomorphisms between polycyclic-by-finite groups, determines whether their Reidemeister number is finite, and additionally returns a set of representatives of the twisted conjugacy classes if it is.

1. Introduction

Let G and H be groups and let $\varphi, \psi \colon H \to G$ be group homomorphisms. Then H acts on G (from the right) by

$$G \times H \to G \colon (g, h) \mapsto \varphi(h)^{-1} g \psi(h).$$

When two elements $g_1, g_2 \in G$ belong to the same orbit under this action, we say they are (φ, ψ)-*twisted conjugate*. These orbits are called the (φ, ψ)-*twisted conjugacy classes*, or the *Reidemeister classes* of the pair (φ, ψ). We denote the (φ, ψ)-twisted conjugacy class of g by $[g]_{\varphi,\psi}$. The set of Reidemeister classes of the pair (φ, ψ) is denoted by $\mathcal{R}[\varphi, \psi]$. The *Reidemeister number* $R(\varphi, \psi)$ is the cardinality of $\mathcal{R}[\varphi, \psi]$ and is either a positive integer or infinity.

The notion of twisted conjugacy finds its origin in topological coincidence theory; we refer to [13] for a survey on the subject. If f and g are two continuous maps between topological spaces X and Y, then the Reidemeister number $R(f_*, g_*)$ of the induced group homomorphisms $f_*, g_* \colon \pi_1(X) \to \pi_1(Y)$ holds information on the least number of coincidence points of (the homotopy classes of) f and g.

One of the key components in extracting information from the Reidemeister number is the ability to determine its finiteness. This has led to the systematic search for groups with the R_∞-property, i.e. groups for which every Reidemeister number $R(\varphi, \mathrm{id})$ (with φ an automorphism) is infinite. Recently, this property has been studied for soluble arithmetic groups [16, 17], right-angled Artin groups [11, 14], braid groups [9, 10], linear groups [18, 19] and generalised Baumslag-Solitar groups [26, 27], to name just a few results.

The algorithmic study of twisted conjugacy has mostly been focused around solving the twisted conjugacy problem, i.e. deciding whether or not two elements of a group are twisted conjugate. Recently, solutions were found for e.g. Artin groups [3, 7, 8] and direct products of free groups [5, 6]. Of particular interest for this paper are the results obtained by Roman'kov for metabelian and polycyclic groups [21–24]. In this paper, we solve the following related problem:

Problem A. Given two polycyclic-by-finite groups G and H and two homomorphisms $\varphi, \psi \colon H \to G$, determine whether $R(\varphi, \psi)$ is finite, and if it is, find a set $\{g_1, g_2, \dots, g_k\} \subseteq G$ such that $G = [g_1]_{\varphi,\psi} \sqcup [g_2]_{\varphi,\psi} \sqcup \dots \sqcup [g_k]_{\varphi,\psi}$.

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In [12], this problem was solved for the case $G = H$. The algorithms obtained in that paper cannot be generalised straightforwardly to obtain a solution to Problem A. Instead, we take a different approach and actually consider a more general problem, for which we first introduce some notation.

Definition 1.1. A quintuple (G, H, φ, ψ, N) where

- G and H are polycyclic-by-finite groups,
- φ and ψ are group homomorphisms $H \to G$,
- N is a normal subgroup of G such that $\varphi(h)^{-1}\psi(h) \in N$ for every $h \in H$,

will be called a *standard quintuple.*

In this situation, the (φ, ψ)-twisted conjugacy action of H on G can be restricted to an action of H on N. We denote the set of orbits of this action by $\mathcal{R}_N[\varphi, \psi]$, and the number of orbits by $R_N(\varphi, \psi)$. The more general problem we aim to solve is the following.

Problem B. Given a standard quintuple (G, H, φ, ψ, N), determine whether or not $R_N(\varphi, \psi)$ is finite, and if it is, find a set $\{n_1, n_2, \ldots, n_k\} \subseteq N$ such that $N = [n_1]_{\varphi,\psi} \sqcup [n_2]_{\varphi,\psi} \sqcup \cdots \sqcup [n_k]_{\varphi,\psi}$.

By taking $N = G$, clearly this reduces to Problem A. The main result of this paper is an explicit algorithm that solves this problem.

Algorithm A. There exists an algorithm that solves Problem B for any standard quintuple (G, H, φ, ψ, N).

It is worth mentioning that while this algorithm is more widely applicable than the one obtained in [12], it is expected to be much less efficient, in terms of both speed and memory. It is the intention of the author to implement Algorithm A as part of the GAP package TwistedConjugacy [29] in the near future.

2. Preliminaries

In this section, we briefly cover some of the properties (algorithmic or otherwise) of polycyclic-by-finite groups that we will make use of in the sequel. For a treatise on polycyclic (and by extension polycyclic-by-finite) groups, we refer to [25].

Proposition 2.1. *Let G be a polycyclic-by-finite group. Then:*

- *G is nilpotent-by-abelian-by-finite,*
- *every subgroup of G is finitely generated,*
- *any ascending subgroup series in G stabilises.*

In particular, if $Z_i(G)$ denotes the i-th term of the upper central series of a polycyclic-by-finite group G, i.e.

$$Z_0(G) = 1, \quad \frac{Z_{i+1}(G)}{Z_i(G)} = Z\left(\frac{G}{Z_i(G)}\right),$$

then there exists a $k \in \mathbb{N}$ such that $Z_k(G) = Z_{k+1}(G)$.

We refer to [2] for a comprehensive treatment of the algorithmic theory of polycyclic-by-finite groups. To summarise some of the results we require, let G and H be polycyclic-by-finite groups, K a subgroup and N a normal subgroup of G, and φ a homomorphism from H to G. Then there exist algorithms that calculate or construct the following:

- the Fitting subgroup $\mathrm{Fitt}(G)$ of G,
- the derived subgroup G' of G,
- the centre $Z(G)$ of G,
- a nilpotent-by-abelian finite index normal subgroup of G,

- the index $[G : K]$,
- the intersection $N \cap K$,
- the product $NK \leq G$,
- the image $\varphi(H)$,
- the preimage $\varphi^{-1}(N)$.

The following two algorithms, which are more specific to our setting, can be found in [23, Section 7] and [1, Section 5.4.6]. Implementations (for polycyclic groups) are available in the GAP package TwistedConjugacy [29].

Algorithm 2.2. Let G and H be polycyclic-by-finite groups, let $\varphi, \psi \in \operatorname{Hom}(H, G)$ and let $g_1, g_2 \in G$. There exists an algorithm that decides whether $[g_1]_{\varphi,\psi} = [g_2]_{\varphi,\psi}$ or not.

Algorithm 2.3. Let G and H be polycyclic-by-finite groups and let $\varphi, \psi \in \operatorname{Hom}(H, G)$. There exists an algorithm that calculates the *coincidence group*

$$\operatorname{Coin}(\varphi, \psi) := \{ h \in H \mid \varphi(h) = \psi(h) \}$$

of the pair (φ, ψ).

For notational convenience, let us introduce a slight generalisation of the coincidence group. Consider the situation from Algorithm 2.3 and let N be a normal subgroup of G. We define

$$\operatorname{Coin}_N(\varphi, \psi) := \{ h \in H \mid \varphi(h)^{-1}\psi(h) \in N \}.$$

Let $p \colon G \to G/N$ be the natural projection and set $\bar{\varphi} := p \circ \varphi$, $\bar{\psi} := p \circ \psi$, then $\operatorname{Coin}_N(\varphi, \psi) = \operatorname{Coin}(\bar{\varphi}, \bar{\psi})$. Thus, any such group can also be calculated using Algorithm 2.3.

Finally, let us fix some notation. We shall work exclusively with right group actions, and use exponents to denote the conjugation action ($x^y := y^{-1}xy$) and square brackets to denote the commutator ($[x, y] := x^{-1}y^{-1}xy$). If G is a group and $g \in G$, then ι_g denotes the inner automorphism of G given by $x \mapsto g^{-1}xg$.

3. Group modules, derivations and cohomology

Up until the final theorem, most of this section consists of a basic introduction to group cohomology, and can be found in most standard works on the subject (e.g. [4]). We denote the (right) action of a group Q on another group A by exponents, i.e. the action of Q on A is given by the map

$$A \times Q \to A \colon (a, q) \mapsto a^q$$

Let us also introduce the notation $[a, q] := a^{-1}a^q$. We remark that this notation coincides with that of the previous section if A and Q are subgroups of a common group and Q acts on A by conjugation.

Definition 3.1. Let Q be a group acting on an abelian group A such that for all $a_1, a_2 \in A$ and all $q \in Q$ we have $(a_1a_2)^q = a_1^q a_2^q$. Then A is called a *Q-module.*

Definition 3.2. Let Q be a group, A a Q-module and B a subgroup of A. If $b^q \in B$ for all $b \in B$ and all $q \in Q$, then B is called a *Q-submodule* of A.

It is an easy exercise to check that the quotient A/B of a Q-module A by a Q-submodule B inherits a Q-module structure from A.

Definition 3.3. Let Q be a group and let A be a Q-module. By A^Q we denote the set of all Q-invariant elements of A, i.e.

$$A^Q := \{ a \in A \mid \forall q \in Q : [a, q] = 1 \}.$$

It is easy to see that A^Q is a Q-submodule of A. In terms of cohomology, this is exactly the zeroth cohomology group $H^0(Q, A)$.

Definition 3.4. Let Q be a group and A a Q-module. A map $\delta\colon Q \to A$ is called a *derivation* (or *crossed homomorphism*) if $\delta(q_1q_2) = \delta(q_1)^{q_2}\delta(q_2)$ for all $q_1, q_2 \in Q$. The set of all derivations from Q to A forms an abelian group (under pointwise addition) and is denoted by $\mathrm{Der}(Q, A)$.

In general, a derivation $\delta\colon Q \to A$ is not a group homomorphism, unless Q acts trivially on the image of δ. Next, we consider the analogue of inner automorphisms in the context of derivations.

Definition 3.5. Let Q be a group and A a Q-module. For any $a \in A$, the map

$$\delta_a\colon Q \to A\colon q \mapsto [a, q]$$

is a derivation. Such derivations are called *inner derivations* or *principal derivations*; the set of all inner derivations from Q to A forms a subgroup of $\mathrm{Der}(Q, A)$ and is denoted by $\mathrm{IDer}(Q, A)$.

Since $\mathrm{Der}(Q, A)$ is abelian, its subgroup $\mathrm{IDer}(Q, A)$ is automatically normal. Thus, we can consider their quotient.

Definition 3.6. The quotient group

$$H^1(Q, A) \coloneqq \frac{\mathrm{Der}(Q, A)}{\mathrm{IDer}(Q, A)}$$

is called the *first cohomology group*.

We now have the necessary background to prove the following theorem, which is the main (and only) result of this section. Its proof is almost directly copied from part (iii) in the proof of [20, Theorem B].

Theorem 3.7. *Let A and Q be finitely generated abelian groups such that A is a Q-module with $A^Q = 1$. If a surjective derivation $\delta\colon Q \to A$ exists, then A is finite.*

Proof. Let $\delta\colon Q \to A$ be a surjective derivation. Applying [15, Theorem H] gives us that $H^1(Q, A)$ has finite exponent, which we denote by m. The derivation

$$\delta^m\colon Q \to A\colon q \mapsto \delta(q)^m$$

is then inner. Let p be a prime not dividing m, note that $A^p = \{\, a^p \mid a \in A \,\}$ is a Q-submodule of A and set $\bar{A} \coloneqq A/A^p$. If $\pi\colon A \to \bar{A}$ is the natural projection, then $\bar{\delta} \coloneqq \pi \circ \delta$ is a surjective derivation and $\bar{\delta}^m$ is inner. But as $H^1(Q, \bar{A})$ is a p-group and p does not divide m, $\bar{\delta}$ itself is inner, so there exists some $\bar{a} \in \bar{A}$ such that $\bar{\delta}(q) = [\bar{a}, q]$ for all $q \in Q$. Because $\bar{\delta}$ is surjective, there exists an $x \in Q$ for which $\bar{\delta}(x) = \bar{a}^{-1}$. Thus

$$\bar{a}^{-1} = \bar{\delta}(x) = [\bar{a}, x] = \bar{a}^{-1}\bar{a}^x.$$

But then $\bar{a}^x = \bar{1}$ and hence $\bar{a} = \bar{1}$, so $\bar{\delta}$ is the trivial derivation. We find that

$$A^p \subseteq A = \delta(Q) \subseteq \pi^{-1}(\bar{\delta}(Q)) = \pi^{-1}(\bar{1}) = A^p,$$

which means that $A = A^p$ and thus A must be finite. □

4. General algorithms

In this section, we provide two "auxiliary" algorithms that will prove to be indispensable in the coming sections. The first algorithm reduces the calculation of twisted conjugacy classes in $N \trianglelefteq G$ to that of twisted conjugacy classes in $N \cap K \trianglelefteq G$ and in $NK/K \trianglelefteq G/K$, for a normal subgroup $K \trianglelefteq G$. The underlying idea is described in the next lemma.

Lemma 4.1. *Let (G, H, φ, ψ, N) be a standard quintuple and let $K \trianglelefteq G$. Let $p\colon G \to G/K$ be the natural projection, set $M \coloneqq N \cap K$ and define*

$$\bar{G} \coloneqq p(G), \qquad \bar{\varphi} \coloneqq p \circ \varphi,$$
$$\bar{N} \coloneqq p(N), \qquad \bar{\psi} \coloneqq p \circ \psi.$$

Then, for any $n \in N$, define

$$C_n \coloneqq \mathrm{Coin}_K(\iota_n \varphi, \psi),$$
$$\varphi_n\colon C_n \to G\colon c \mapsto (\iota_n \varphi)(c), \qquad \psi_n\colon C_n \to G\colon c \mapsto \psi(c).$$

Let $n_1, n_2, \ldots \in N$ such that $\bar{N} = \bigsqcup_i [p(n_i)]_{\bar{\varphi},\bar{\psi}}$, and for each n_i, let $m_{n_i 1}, m_{n_i 2}, \ldots \in M$ such that $M = \bigsqcup_j [m_{n_i j}]_{\varphi_{n_i}, \psi_{n_i}}$. Then N is the (disjoint) union of the following (φ, ψ)-twisted conjugacy classes:

$$N = \bigsqcup_i \bigsqcup_j [n_i m_{n_i j}]_{\varphi,\psi}.$$

In particular, $R_N(\varphi, \psi)$ is finite if and only if $R_{\bar{N}}(\bar{\varphi}, \bar{\psi})$ is finite and $R_M(\varphi_{n_i}, \psi_{n_i})$ is finite for every n_i.

Proof. It suffices to prove that every $n \in N$ belongs to exactly one twisted conjugacy class $[n_i m_{ij}]_{\varphi,\psi}$. So let $n \in N$ and consider $\bar{n} \coloneqq p(n)$. Let $i \in \mathbb{N}$ such that $[\bar{n}]_{\bar{\varphi},\bar{\psi}} = [\bar{n}_i]_{\bar{\varphi},\bar{\psi}}$. There exists an $h_1 \in H$ such that

$$\bar{n} = \bar{\varphi}(h_1)^{-1} \bar{n}_i \bar{\psi}(h_1).$$

Lifting this back to G, there exists an $m \in M$ for which

$$n = \varphi(h_1)^{-1} n_i m \psi(h_1) = n_i (\iota_{n_i} \varphi)(h_1)^{-1} m \psi(h_1).$$

Now let $j \in \mathbb{N}$ such that $[m]_{\varphi_i,\psi_i} = [m_{n_i j}]_{\varphi_i,\psi_i}$. There exists an $h_2 \in H$ such that

$$m = \varphi_i(h_2)^{-1} m_{n_i j} \psi_i(h_2) = (\iota_{n_i} \varphi)(h_2)^{-1} m_{n_i j} \psi(h_2).$$

Combining the previous two equations we obtain that

$$n = n_i (\iota_{n_i} \varphi)(h_2 h_1)^{-1} m_{n_i j} \psi(h_2 h_1) = \varphi(h_2 h_1)^{-1} n_i m_{n_i j} \psi(h_2 h_1),$$

hence indeed $n \in [n_i m_{n_i j}]_{\varphi,\psi}$ for some $i, j \in \mathbb{N}$. We omit the (straightforward) calculation that this union is indeed disjoint. □

Algorithm 4.2. Let (G, H, φ, ψ, N) be a standard quintuple and let $K \trianglelefteq G$. Using the definitions from Lemma 4.1, suppose that there exists an algorithm that solves Problem B for $(\bar{G}, H, \bar{\varphi}, \bar{\psi}, \bar{N})$, and for any $n \in N$ there exists an algorithm that does the same for $(G, C_n, \varphi_n, \psi_n, M)$. Then there exists an algorithm that solves Problem B for (G, H, φ, ψ, N).

One limitation of this algorithm is that the group G remains unchanged when we pass from N to $M = N \cap K$. By contrast, the second algorithm gives us a way to pass from G to a finite index normal subgroup of G.

Algorithm 4.3. Let (G, H, φ, ψ, N) be a standard quintuple and let $K \trianglelefteq G$ with $[G : K] < \infty$. Suppose that there exists an algorithm that solves Problem B for any standard quintuple of the form (K, L, λ, μ, M). There exists an algorithm that solves Problem B for (G, H, φ, ψ, N).

Proof. We consider two cases. First, suppose that N is contained in K. Define

$$L \coloneqq \varphi^{-1}(K) \cap \psi^{-1}(K),$$
$$\lambda\colon L \to K\colon l \mapsto \varphi(l), \qquad \mu\colon L \to K\colon l \mapsto \psi(l),$$

and consider the (surjective) map

$$\pi\colon \mathcal{R}_N[\lambda,\mu] \to \mathcal{R}_N[\varphi,\psi]\colon [n]_{\lambda,\mu} \mapsto [n]_{\varphi,\psi}.$$

Since K is a finite index normal subgroup of G, L is a finite index normal subgroup of H, and thus we can pick a transversal $\{h_1,\dots,h_r\}$ of L in H. Now suppose that $n,n' \in N$ are elements such that $[n]_{\varphi,\psi} = [n']_{\varphi,\psi}$, i.e. there exists some $h \in H$ such that $n' = \varphi(h)^{-1}n\psi(h)$. For some (unique!) h_i and some $l \in L$, we have $h = h_i l$. But then

$$n' = \varphi(h_i l)^{-1}n\psi(h_i l) = \lambda(l)^{-1}\varphi(h_i)^{-1}n\psi(h_i)\mu(l),$$

which gives us that

$$[n']_{\lambda,\mu} = [\varphi(h_i)^{-1}n\psi(h_i)]_{\lambda,\mu}.$$

So for any $n \in N$, the preimage $\pi^{-1}([n]_{\varphi,\psi})$ is finite, and in particular $R_N(\lambda,\mu) < \infty$ if and only if $R_N(\varphi,\psi) < \infty$. If $R_N(\lambda,\mu)$ is infinite, then so is $R_N(\varphi,\psi)$ and the algorithm finishes. Otherwise, we obtain a set $\{n_1,n_2,\dots,n_t\} \subseteq N$ such that $N = [n_1]_{\lambda,\mu} \sqcup [n_2]_{\lambda,\mu} \sqcup \dots \sqcup [n_t]_{\lambda,\mu}$.

Applying Algorithm 2.2 to pairs of n_i's, we can reduce this to a finite set $\{n_{i_1},n_{i_2},\dots,n_{i_s}\}$ such that $N = [n_{i_1}]_{\varphi,\psi} \sqcup [n_{i_2}]_{\varphi,\psi} \sqcup \dots \sqcup [n_{i_s}]_{\varphi,\psi}$, which finishes the first case.

We now move on to the second case: N is not contained in K. We apply Algorithm 4.2 for the normal subgroup K. Indeed, $\bar{G}$ is finite, which poses no problem, and for each of the quintuples $(G,C_{n_i},\varphi_{n_i},\psi_{n_i},M)$ we note that $M \leq K$, hence we can repeat the steps described in the first case. □

5. Nilpotent-by-finite groups

The main result of this section is the existence of the algorithm below.

Algorithm 5.1. There exists an algorithm that solves Problem B for any standard quintuple (G,H,φ,ψ,N) where G is nilpotent-by-finite.

Rather than providing one large algorithm immediately, we split this up in several "sub-algorithms". A first step is to consider the case where N is a central subgroup of G.

Algorithm 5.2. There exists an algorithm that solves Problem B for any standard quintuple (G,H,φ,ψ,N) where N is central.

Proof. We start by constructing the group homomorphism given by

$$\delta\colon H \to N\colon \varphi(h)^{-1}\psi(h).$$

Let $p\colon N \to N/\delta(H)$ be the natural projection and set $\bar{N} := p(N)$. Since N is central, it follows that the map

$$\pi\colon \mathcal{R}_N[\varphi,\psi] \to \bar{N}\colon [n]_{\varphi,\psi} \mapsto p(n)$$

is a bijection. Thus, if $\bar{N}$ is infinite, so is $R_N(\varphi,\psi)$. Otherwise, denote the elements of $\bar{N}$ by $\bar{n}_1,\dots,\bar{n}_k$. For each $i \in \{1,\dots,k\}$, pick an element $n_i \in p^{-1}(\bar{n}_i)$. Then the set $\{n_1,\dots,n_k\}$ satisfies $N = \bigsqcup_{i=1}^k [n_i]_{\varphi,\psi}$, so we have solved Problem B. □

A nilpotent group is a group such that for some $k \in \mathbb{N}$, $Z_k(G) = G$. This hints at the possibility of tackling the case where G is nilpotent by inductively applying the previous algorithm.

Algorithm 5.3. There exists an algorithm that solves Problem B for any standard quintuple (G,H,φ,ψ,N) where G is nilpotent.

Proof. Let (G, H, φ, ψ, N) be a standard quintuple with G nilpotent of class c. We prove this by induction on c. If $c = 0$, then G is trivial and therefore $N = [1]_{\varphi,\psi}$.

Now suppose that $c > 0$. We can apply Algorithm 4.2 for the normal subgroup $K := Z(G)$. Indeed, if we consider the quintuple $(\bar{G}, H, \bar{\varphi}, \bar{\psi}, \bar{N})$, then $\bar{G}$ is nilpotent of class $c - 1$, so the required algorithm exists by induction; for any $n \in N$ the quintuple $(G, C_n, \varphi_n, \psi_n, M)$ has $M \leq Z(G)$, so we can apply Algorithm 5.2. □

Finally, the general case of G being nilpotent-by-finite now follows easily by combining previously obtained algorithms.

Proof of Algorithm 5.1. Let K be the Fitting subgroup of G. Since Algorithm 5.3 can solve Problem B for any standard quintuple of the form (K, L, λ, μ, M), the result follows from Algorithm 4.3. □

6. Metabelian groups

Like the preceding section, the main result here is an algorithm solving Problem B when some additional conditions are placed on the standard quintuple. In particular, G will be metabelian throughout this section.

Algorithm 6.1. There exists an algorithm that solves Problem B for any standard quintuple (G, H, φ, ψ, A) where

(1) H is abelian,
(2) A is abelian,
(3) $G = A\varphi(H)$.

Such quintuple will be called a *metabelian quintuple.*

We construct this algorithm with the help of some theoretical results, which come in the form of the next three lemmas.

Lemma 6.2. *Let (G, H, φ, ψ, A) be a metabelian quintuple such that $R_A(\varphi, \psi) = 1$ and $Z(G) = 1$. Then A is finite.*

Proof. The group H acts on A via

$$A \times H \to A\colon (a, h) \mapsto a^h := \varphi(h)^{-1} a \varphi(h),$$

so A is an H-module. Since H is abelian, the map δ defined by

$$\delta\colon H \to A\colon \varphi(h)^{-1}\psi(h)$$

is a derivation, and as $A = [1]_{\varphi,\psi} = \delta(H)$ it is surjective. Now suppose $a \in A^H$, i.e. $1 = [a, h] = [a, \varphi(h)]$ for all $h \in H$. Since $G = A\varphi(H)$, it then follows that $a \in Z(G)$ and therefore $a = 1$, so A^H is trivial. By Theorem 3.7, A is finite. □

Lemma 6.3. *Let (G, H, φ, ψ, A) be a metabelian quintuple such that $R_A(\varphi, \psi) = 1$. Then G is nilpotent-by-finite.*

Proof. As G is polycyclic, its upper central series eventually stabilises, so for some $k \in \mathbb{N}$ we have $Z_k(G) = Z_{k+1}(G)$. Let $p\colon G \to G/Z_k(G)$ be the natural projection and set $\bar{G} := p(G)$, $\bar{A} := p(A)$, $\bar{\varphi} := p \circ \varphi$ and $\bar{\psi} := p \circ \psi$.

Then $(\bar{G}, H, \bar{\varphi}, \bar{\psi}, \bar{A})$ is a metabelian quintuple that satisfies the conditions of Lemma 6.2, hence $\bar{A}$ is finite and $\bar{G}$ is therefore finite-by-abelian. It follows from [28, Lemma 6.3] that $\bar{G}$ is then abelian-by-finite. Now let $\bar{B}$ be an abelian finite index normal subgroup of $\bar{G}$, set $B := p^{-1}(\bar{B})$ and set $\bar{Z} := p(Z_k(B))$. Since $Z_k(G) \leq B$, we also have that $Z_i(G) \leq Z_i(B)$ for all $i \in \mathbb{N}$. From the third isomorphism theorem, we find that

$$\frac{B}{Z_k(B)} \cong \frac{B/Z_k(G)}{Z_k(B)/Z_k(G)} = \frac{\bar{B}}{\bar{Z}}.$$

Since $B/Z_k(B)$ is isomorphic to a quotient of the abelian group $\bar{B}$, it is itself abelian. But then $Z_{k+1}(B) = B$, hence B is nilpotent. And as B has finite index in G, the latter is indeed nilpotent-by-finite. $\square$

Lemma 6.4. *Let (G, H, φ, ψ, A) be a metabelian quintuple such that $R_A(\varphi, \psi) < \infty$. Then G is nilpotent-by-finite.*

Proof. Let $a_1 = 1, a_2, \dots, a_n \in A$ such that $A = \bigsqcup_{i=1}^n [a_i]_{\varphi,\psi}$. Invoking [28, Thm. 5.1], for each $i \in \{2, \dots, n\}$, there exists a finite index normal subgroup N_i of G such that $a_i \notin [1]_{\varphi,\psi} N_i$. Define $N := \cap_{i=2}^n N_i$, then $B := A \cap N$ has finite index in A, is normal in G and is a subset of $[1]_{\varphi,\psi}$. Next, we set $C := \mathrm{Coin}_N(\varphi, \psi)$, $K := B\,\varphi(C)$, and we define $\lambda, \mu \in \mathrm{Hom}(C, K)$ by

$$\lambda \colon C \to K \colon c \mapsto \varphi(c), \quad \mu \colon C \to K \colon c \mapsto \psi(c).$$

It is straightforward to verify that the quintuple (K, C, λ, μ, B) satisfies the conditions of Lemma 6.3, so K is nilpotent-by-finite. As $[G : N] < \infty$, we find that $[H : C] < \infty$, next that $[\varphi(H) : \varphi(C)] < \infty$, and finally that

$$[G : K] = [A\,\varphi(H) : B\,\varphi(C)] < \infty,$$

so G is nilpotent-by-finite as well. $\square$

With this final lemma proven, we are ready to construct the required algorithm. However, since we already proved the existence of an algorithm that solves Problem B when G is nilpotent-by-finite in the previous section, this takes very little work.

Proof of Algorithm 6.1. We first calculate whether or not G is nilpotent-by-finite, by constructing the Fitting subgroup of G and then checking whether its index in G is finite or not. If it is not, then by the contrapositive of Lemma 6.4 $R_A(\varphi, \psi) = \infty$. If it is, then we defer to Algorithm 5.1. $\square$

7. Abelian subgroup commuting with the derived subgroup

Once again, the algorithm below is the main result of this section. The additional conditions we place on the standard quintuple here are essentially the same conditions Roman'kov used in his proofs of Algorithms 2.2 and 2.3, see [21, 23].

Algorithm 7.1. There exists an algorithm that solves Problem B for any standard quintuple (G, H, φ, ψ, A) where

(1) A is abelian,
(2) $[A, G'] = 1$.

Such quintuple will be called an *ABCD-quintuple.*

To start off we impose two additional conditions on the quintuple, which allow us to reduce the problem to that of a metabelian quintuple and hence apply the results from Section 6.

Algorithm 7.2. There exists an algorithm that solves Problem B for any ABCD-quintuple (G, H, φ, ψ, A) where $G = A\,\varphi(H)$ and $H' \leq \mathrm{Coin}(\varphi, \psi)$.

Proof. Let $p \colon H \to H^{\mathrm{ab}}$ be the natural projection. Since $[A, G'] = 1$, the map

$$A \times H^{\mathrm{ab}} \to A \colon (a, \bar{h}) \mapsto a^{\bar{h}} := \varphi(h)^{-1} a \varphi(h),$$

with $h \in p^{-1}(\bar{h})$, defines a well-defined action of H^{ab} on A. Construct the semidirect product $S := A \rtimes H^{\mathrm{ab}}$ and define the group homomorphisms

$$\lambda \colon H^{\mathrm{ab}} \to S \colon \bar{h} \mapsto (1, \bar{h}),$$
$$\mu \colon H^{\mathrm{ab}} \to S \colon \bar{h} \mapsto (\varphi(h)^{-1}\psi(h), \bar{h}),$$

where again $h \in p^{-1}(\bar{h})$. Then $(S, H^{\mathrm{ab}}, \lambda, \mu, A)$ is a metabelian quintuple and the map

$$\mathcal{R}_A[\varphi, \psi] \to \mathcal{R}_A[\lambda, \mu] \colon [a]_{\varphi,\psi} \mapsto [a]_{\lambda,\mu}$$

is a bijection, so it now suffices to apply Algorithm 6.1. □

Next, we eliminate the condition $H' \leq \mathrm{Coin}(\varphi, \psi)$ from the above algorithm.

Algorithm 7.3. There exists an algorithm that solves Problem B for any ABCD-quintuple (G, H, φ, ψ, A) where $G = A\,\varphi(H)$.

Proof. Define the map

$$\delta \colon H' \to A \colon h \mapsto \varphi(h)^{-1}\psi(h),$$

which is a group homomorphism since $[A, G'] = 1$. It follows from the condition $G = A\,\varphi(H)$ that the image $\delta(H')$ is a normal subgroup of G. Let $p \colon G \to G/\delta(H')$ be the natural projection and set $\bar{G} := p(G)$, $\bar{A} := p(A)$, $\bar{\varphi} := p \circ \varphi$ and $\bar{\psi} := p \circ \psi$. Then the map

$$\mathcal{R}_A[\varphi, \psi] \to \mathcal{R}_{\bar{A}}[\bar{\varphi}, \bar{\psi}] \colon [a]_{\varphi,\psi} \mapsto [p(a)]_{\bar{\varphi},\bar{\psi}}$$

is a bijection. It then suffices to solve Problem B for $(\bar{G}, H, \bar{\varphi}, \bar{\psi}, \bar{A})$, which is an ABCD-quintuple satisfying the requirements of Algorithm 7.2. □

And finally, we eliminate the condition $G = A\,\varphi(H)$ from the above algorithm.

Proof of Algorithm 7.1. Set $K := A\,\varphi(H)$ and define

$$\lambda \colon H \to K \colon h \mapsto \varphi(h), \quad \mu \colon H \to K \colon h \mapsto \psi(h),$$

then (K, H, λ, μ, A) is an ABCD-quintuple with $K = A\lambda(H)$. Then the map

$$\mathcal{R}_A[\varphi, \psi] \to \mathcal{R}_A[\lambda, \mu] \colon [a]_{\varphi,\psi} \mapsto [a]_{\lambda,\mu}$$

is bijective, so it suffices to apply Algorithm 7.3 to (K, H, λ, μ, A). □

8. General case

With most of the preliminary algorithms now at our disposal, we are ready to tackle Problem B in full generality.

Algorithm 8.1. There exists an algorithm that solves Problem B for any standard quintuple (G, H, φ, ψ, N) where G is nilpotent-by-abelian.

Proof. Since G is nilpotent-by-abelian, its derived subgroup G' is nilpotent, say of class c. We proceed by induction on c. If $c = 0$, then G is abelian and N is central, hence we defer to Algorithm 5.2.

Now suppose that $c > 0$. Just like in the proof of Algorithm 5.3, we will apply Algorithm 4.2, but for the subgroup $K := Z(G')$. If we consider the quintuple $(\bar{G}, H, \bar{\varphi}, \bar{\psi}, \bar{N})$, then $\bar{G}$ is nilpotent-by-abelian with $\bar{G}'$ nilpotent of class $c - 1$, so the required algorithm exists by induction. And for any $n \in N$, the quintuple $(G, C_n, \varphi_n, \psi_n, M)$ has $M = K \cap N$ abelian and $[M, G'] \leq [K, G'] = 1$, hence it is an ABCD-quintuple and we can apply Algorithm 7.1. □

Proof of Algorithm A. Let K be a nilpotent-by-abelian finite index normal subgroup of G. Algorithm 8.1 can solve Problem B for any standard quintuple of the form (K, L, λ, μ, M), hence the result follows from Algorithm 4.3. □

9. Derivations to non-abelian groups

A crucial component in the construction of Algorithm A was Theorem 3.7, which was extracted from the proof of Theorem B in [20]. For the reader's convenience, we give the statement of the referenced theorem.

Algorithm 9.1. There is an algorithm which, when given a polycyclic-by-finite group Q, a finitely generated abelian group A with an explicit Q-module structure, and a derivation $\delta\colon Q \to A$, decides if δ is surjective.

Using Algorithm A, we can generalise this result to "non-abelian modules", which we define below.

Definition 9.2. Let Q be a group acting on a group G such that for all $g_1, g_2 \in G$ and all $q \in Q$ we have $(g_1 g_2)^q = g_1^q g_2^q$. Then G is called a *Q-group.*

The notion of derivation naturally extends to Q-groups, although it should be mentioned that the set of all derivations from a group Q to a Q-group G need not be a group under the pointwise multiplication, in contrast to the abelian case. We can now extend Algorithm 9.1 in the following way:

Algorithm B. There is an algorithm which, when given two polycyclic-by-finite groups Q and G, with G having an explicit Q-group structure, and a derivation $\delta\colon Q \to G$, decides if δ is surjective.

Proof. We construct the semi-direct product $G \rtimes Q$ and the two homomorphisms

$$\varphi\colon Q \to G \rtimes Q\colon q \mapsto (1, q),$$
$$\psi\colon Q \to G \rtimes Q\colon q \mapsto (\delta(q), q).$$

Because $\delta(Q) = [1]_{\varphi,\psi} \subseteq G$, δ is surjective if and only if $R_G(\varphi, \psi) = 1$. So we can apply Algorithm A to the standard quintuple $(G \rtimes Q, Q, \varphi, \psi, G)$. If the algorithm returns a singleton $\{g\} \subseteq G$, then δ is surjective. If it returns a finite set $\{g_1, \dots, g_n\}$ with $n > 1$ or it determines that $R_G(\varphi, \psi) = \infty$, then δ is not surjective. $\square$

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References

[1] F. Bassino et al. *Complexity and Randomness in Group Theory.* De Gruyter, 2020. xii+374. DOI: 10/grwd34.
[2] G. Baumslag, F. B. Cannonito, D. J. S. Robinson and D. Segal. 'The algorithmic theory of polycyclic-by-finite groups'. In: *J. Algebra* 142.1 (1991), pp. 118–149. DOI: 10/dxh5cq.
[3] M. Blufstein and M. Valiunas. 'On the twisted conjugacy problem for large-type Artin groups'. 2024. arXiv: 2411.05493 [math.GR].
[4] K. S. Brown. *Cohomology of Groups.* Vol. 87. Grad. Texts Math. Springer, 1982. x+309.
[5] A. Carvalho. 'On the dynamics of endomorphisms of the direct product of two free groups'. 2023. arXiv: 2307.13875 [math.GR].
[6] A. Carvalho and J. Delgado. 'Orbit problems for free-abelian times free groups and related families'. In: *J. Algebra Appl.*, 2650002 (2024). Advance Online Publication. DOI: 10/g9bnnd.
[7] G. Crowe. 'Twisted conjugacy in dihedral Artin groups I: Torus Knot groups'. 2024. arXiv: 2403.16671 [math.GR].

[8] G. Crowe. 'Twisted conjugacy in dihedral Artin groups II: Baumslag Solitar groups $\mathrm{BS}(n, n)$'. 2024. arXiv: `2404.04705 [math.GR]`.

[9] K. Dekimpe, D. L. Gonçalves and O. Ocampo. 'The R_∞ property for pure Artin braid groups'. In: *Monatsh. Math.* 195.1 (2021), pp. 15–33. DOI: `10/grv3jk`.

[10] K. Dekimpe, D. L. Gonçalves and O. Ocampo. 'Characteristic subgroups and the R_∞-property for virtual braid groups'. In: *J. Algebra* 663 (2025), pp. 20–47. DOI: `10/g898jp`.

[11] K. Dekimpe and P. Senden. 'The R_∞-property for right-angled Artin groups'. In: *Topology Appl.* 293, 107557 (2021). DOI: `10/grv3jm`.

[12] K. Dekimpe and S. Tertooy. 'Algorithms for twisted conjugacy classes of polycyclic-by-finite groups'. In: *Topology Appl.* 293, 107565 (2021). DOI: `10/grvpwc`.

[13] D. L. Gonçalves. 'Coincidence Theory'. In: *Handbook of Topological Fixed Point Theory.* Springer, 2005, pp. 3–42. DOI: `10/fr2qnf`.

[14] M. Lathouwers and T. Witdouck. 'The R_∞-property and commensurability for nilpotent groups'. In: *Math. Nachr.* 298.2 (2025), pp. 602–616. DOI: `10/g898jk`.

[15] J. C. Lennox and D. J. S. Robinson. 'Soluble products of nilpotent groups'. In: *Rend. Sem. Mat. Univ. Padova* 62 (1980), pp. 261–280.

[16] P. M. Lins de Araujo and Y. Santos Rego. 'Twisted conjugacy in soluble arithmetic groups'. In: *Math. Nachr.* 298.3 (2024), pp. 763–793. DOI: `10/g898h3`.

[17] P. M. Lins de Araujo and Y. Santos Rego. 'Reidemeister numbers for arithmetic Borel subgroups in type A'. 2023. arXiv: `2306.02936 [math.GR]`.

[18] O. Mitra and P. Sankaran. 'Twisted conjugacy in GL_2 and SL_2 over polynomial algebras over finite fields'. In: *Geom. Dedicata* 216.2, 21 (2022). DOI: `10/gp45ds`.

[19] O. Mitra and P. Sankaran. 'Twisted conjugacy in SL_n and GL_n over subrings of $\overline{\mathbb{F}_p}(t)$'. In: *Groups Geom. Dyn.* 18.3 (2024), pp. 939–962. DOI: `10/g898jh`.

[20] D. J. S. Robinson. 'Derivations and the permutability of subgroups in polycyclic-by-finite groups'. In: *Proc. Amer. Math. Soc.* 130.12 (2002), pp. 3461–3464. DOI: `10/cgnpb8`.

[21] V. A. Roman'kov. 'The twisted conjugacy problem for endomorphisms of polycyclic groups'. In: *J. Group Theory* 13.3 (2010), pp. 355–364. DOI: `10/dtg4x6`.

[22] V. A. Roman'kov. 'On solvability of equations with endomorphisms in nilpotent groups'. In: *Sib. Elektron. Mat. Izv.* 13 (2016), pp. 716–725. DOI: `10/grv3bv`.

[23] V. A. Roman'kov. 'Algorithmic theory of solvable groups'. In: *Prikl. Diskretn. Mat.* 52 (2021), pp. 16–64. DOI: `10/grwdfp`.

[24] V. A. Roman'kov and E. Ventura. 'The twisted conjugacy problem for endomorphisms of metabelian groups'. In: *Algebra Log.* 48.2 (2009), pp. 89–98. DOI: `10/fcg4mr`.

[25] D. Segal. *Polycyclic Groups.* Cambridge Tracts in Mathematics 82. Cambridge University Press, 1983. xiv+289. DOI: `10/fn22fw`.

[26] W. Sgobbi and P. Wong. 'The BNS invariants of the generalized solvable Baumslag-Solitar groups and of their finite index subgroups'. In: *Commun. Algebra* 51.8 (2023), pp. 3354–3370. DOI: `10/g898xv`.

[27] W. C. Sgobbi, D. C. Silva and D. Vendrúscolo. 'The R_∞ property for nilpotent quotients of Generalized Solvable Baumslag-Solitar groups'. In: *J. Group Theory* 26.4 (2023), pp. 725–739. DOI: `10/gr5h55`.

[28] S. Tertooy. 'Twisted conjugacy and separability'. In: *J. Group Theory* (2024). Advance Online Publication. DOI: `10/nv2p`.

[29] S. Tertooy. *TwistedConjugacy. Computation with twisted conjugacy classes.* GAP Package. Version 2.4.0. 2024. URL: `https://stertooy.github.io/TwistedConjugacy`.

KU Leuven Kulak Kortrijk Campus, E. Sabbelaan 53, 8500 Kortrijk, Belgium
Email address: `sam.tertooy@kuleuven.be`
URL: `https://stertooy.github.io`