CLASS NUMBERS AND NILPOTENT SUBGROUPS OF PGL$(2, q)$

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Abstract. We show that for certain odd prime powers q, the number of conjugacy classes of $\mathrm{PGL}(2, q)$ is greater than the order of its largest nilpotent subgroup. This answers negatively a question of Liebeck and Pyber.

1. Introduction

For a finite group G, let $k(G)$ denote its class number (i.e. the number of conjugacy classes) and let $n(G)$ denote the order of its largest nilpotent subgroup. In [6], Liebeck and Pyber prove the existence of a constant $c < 58/21$ such that the inequality $k(G) \leq n(G)^c$ holds for every finite group G, and ask whether this holds with $c = 1$. This is listed as question 14.54 in the Kourovka Notebook [4].

There are many examples of finite groups for which $k(G) = n(G)$ holds, including all finite abelian groups, all holomorphs $C_p \rtimes C_{p-1}$ for p prime, and all projective special linear groups $\mathrm{PSL}(2, q)$ with q even. By analysing their nilpotent subgroups, we prove that for $G := \mathrm{PGL}(2, q)$ with $q \neq 2^r + \epsilon$ for any $r \in \mathbb{N}$ and $\epsilon \in \{-1, 0, 1\}$, $k(G) = q + 2 > q + 1 = n(G)$. As a consequence, $c \geq \log_{12} 13 > 1$. More examples of groups with $k(G) > n(G)$ can then be constructed using these projective linear groups.

2. Nilpotent subgroups of PGL$(2, q)$

In this section, we determine $n(\mathrm{PGL}(2, q))$ for all prime powers q. The subgroups of $\mathrm{PSL}(2, q)$ have been classified in e.g. [5, Thm. 2.1]. For q even, $\mathrm{PGL}(2, q)$ equals $\mathrm{PSL}(2, q)$, so we already know its subgroups. For q odd, $\mathrm{PGL}(2, q)$ contains $\mathrm{PSL}(2, q)$ as an index 2 subgroup, and the former's subgroups can be obtained from the latter's. The resulting classification can be found in e.g. [3, Thm. 2]. We summarise these results below.

Theorem 2.1. *Let $q = p^n$ for p prime. The subgroups of* $\mathrm{PGL}(2, q)$ *are:*

- *(i) Abelian groups:*
  - *(a) C_2,*
  - *(b) C_d, for every $d > 2$ with $d \mid q \pm 1$,*
  - *(c) $(C_2)^2$, unless $q = 2$,*
  - *(d) $(C_p)^m$, for every $m \leq n$.*
- *(ii) Non-abelian dihedral groups:*
  - *(a) D_{2p} (when $p > 2$),*
  - *(b) D_{2d}, for every $d > 2$ with $2d \mid q \pm 1$,*
  - *(c) D_{2d}, for every $d > 2$ with $d \mid q \pm 1$ and $(q \pm 1)/d$ odd.*
- *(iii) Certain non-abelian symmetric and alternating groups.*
- *(iv)* $\mathrm{PSL}(2, p^m)$ *and* $\mathrm{PGL}(2, p^m)$*, for every $m \mid n$.*
- *(v) Non-abelian semi-direct products $(C_p)^m \rtimes C_d$ for every $m \leq n$ and $d \geq 2$ with $d \mid p^{\gcd(m,n)} - 1$.*

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Using this classification, we can state the following proposition.

Proposition 2.2. *The largest nilpotent subgroup of* $G := \mathrm{PGL}(2, q)$ *has order*

$$n(G) = \begin{cases} q+1 & \text{if } q \neq 2^r \pm 1, \\ 2^{r+1} & \text{if } q = 2^r \pm 1, \end{cases}$$

where $r \geq 2$.

Proof. Family (i) consists entirely of abelian groups. The largest group in this family is C_{q+1}, and in the case $q = 3$ it is joint largest with $(C_2)^2$. In any case, the largest order is always $q + 1$.

For family (ii), recall that a dihedral group is nilpotent if and only if its order is a power of 2. Thus, the largest nilpotent dihedral subgroup has order $2d$, where d is the largest power of 2 dividing $q \pm 1$. If $q = 2^r \pm 1$ this subgroup has order $2(q \mp 1) = 2^{r+1}$, otherwise, it has order at most $\frac{2}{3}(q + 1)$.

Families (iii), (iv) and (v) never contain nilpotent groups. For families (iii) and (iv) this is easily checked. For family (v), recall that a finite group is nilpotent if and only if it is a direct product of its Sylow subgroups. But here $(C_p)^m$ and C_d do not commute. □

3. Groups with $k(G) > n(G)$

In [7, Tbl. 3], Macdonald shows that for $G = \mathrm{PGL}(2, q)$, one has $k(G) = q + \gcd(2, q - 1)$. Combining this with Proposition 2.2 we may conclude the following:

Theorem 3.1. *Let* $G = \mathrm{PGL}(2, q)$.

(1) If $q = 2^r$ *with* $r \geq 1$, *then* $k(G) = q + 1 = n(G)$.
(2) If $q = 2^r \pm 1$ *with* $r \geq 2$, *then* $k(G) = q + 2 < 2^{r+1} = n(G)$.
(3) If $q \neq 2^r \pm \epsilon$ *with* $\epsilon \in \{-1, 0, 1\}$, *then* $k(G) = q + 2 > q + 1 = n(G)$.

Thus $k(G) > n(G)$ *if and only if* $q \neq 2^r + \epsilon$ *for any* $r \in \mathbb{N}$ *and* $\epsilon \in \{-1, 0, 1\}$.

Corollary 3.2. *Let* $c \in \mathbb{R}$ *be such that for every finite group* G, $k(G) \leq n(G)^c$. *Then* $c \geq \log_{12} 13 > 1$.

The two smallest groups for which $k(G) > n(G)$ are $\mathrm{PGL}(2, 11)$ (order 1320, class number 13) and $\mathrm{PGL}(2, 13)$ (order 2184, class number 15). Using the GAP [9] packages SmallGrp [2], GrpConst [1] and SmallClassNr [8], we verified that no other examples exist among the groups of order $|G| \leq 2303$, nor among the groups of class number $k(G) \leq 14$.

More examples can be created from these projective linear groups. A first way to do so is by using direct products, since both $k(.)$ and $n(.)$ behave multiplicatively in this case. Taking the direct product of any finite group with sufficiently many copies of $\mathrm{PGL}(2, q)$ ($q \neq 2^r \pm \epsilon$) will result in a group G satisfying $k(G) > n(G)$. For example, $H := \mathrm{SL}(2, 3)$ has $k(H) = 7 < 8 = n(H)$, but the direct product $G := H \times (\mathrm{PGL}(2, 11))^2$ has $k(G) = 1183 > 1152 = n(G)$.

A second way to construct new examples is by taking subdirect products, making use of the fact that $\mathrm{PGL}(2, q)$ with q odd contains $\mathrm{PSL}(2, q)$ as an index 2 (hence normal) subgroup.

Corollary 3.3. *Let* G *be a subdirect product* $P \times_{C_2} A$ *where* $P := \mathrm{PGL}(2, q)$ *with* q *odd and* A *is an abelian group of even order. Then* $k(G) > n(G)$ *if and only if* $q \neq 2^r \pm 1$.

Proof. Let $\pi_1 \colon P \to C_2$ and $\pi_2 \colon A \to C_2$ be the epimorphisms such that

$$G = \{ (x, y) \in P \times A \mid \pi_1(x) = \pi_2(y) \},$$

and set $K_i := \ker(\pi_i)$. If N is the largest nilpotent subgroup in $\mathrm{PGL}(2,q)$, then $N \times A$ is the largest nilpotent subgroup of $P \times A$ and $G \cap (N \times A)$ is the largest nilpotent subgroup of G. Setting $M := N \cap K_1$, we see that

$$G \cap (N \times A) = (M \times K_2) \sqcup (N \backslash M \times A \backslash K_2),$$

so this subgroup has order $|N||A|/2$.

For any subset $X \subseteq P$, let $k_P(X)$ denote the number of P-conjugacy classes contained in X. Since elements of G are conjugate in G if and only if they are conjugate in $P \times A$, we can similarly conclude that

$$k(G) = k_{P\times A}(G) = k_P(K_1)\,|K_2| + k_P(P\backslash K_1)\,|A\backslash K_2| = k(P)\,|A|\,/2.$$

In conclusion, $n(G) = n(P)\,|A|\,/2$ and $k(G) = k(P)\,|A|\,/2$. The result now follows immediately from Theorem 3.1. □

Remark 3.4. For $G := \mathrm{GL}(2,q)$, it is known [7, Tbl. 1] that $k(G) = q^2 - 1$. Moreover, since $\mathrm{PGL}(2,q)$ is the quotient of $\mathrm{GL}(2,q)$ by its centre (which has order $q-1$), the largest nilpotent subgroup of $\mathrm{GL}(2,q)$ is the preimage of the largest nilpotent subgroup of $\mathrm{PGL}(2,q)$ under the quotient map. Thus $n(G) = q^2 - 1 = k(G)$ if and only if $q \neq 2^r \pm 1$.

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