SmallClassNr

Library of groups with small class number

1.2.2

5 December 2024

Sam Tertooy

Sam Tertooy
Email: sam.tertooy@kuleuven.be
Homepage: https://stertooy.github.io/
Address: Wiskunde
KU Leuven Kulak Kortrijk Campus
Etienne Sabbelaan 53
8500 Kortrijk
Belgium

Abstract

The SMALLCLASSNR package provides access to finite groups with small class number. Currently, the package contains the finite groups of class number at most 14.

Copyright

Acknowledgements

This documentation was created using the GAPDOC and AUTODOC packages.

Contents

1 Preface **4**

2 The Small Class Number Library **5**
2.1 Functions 5

References **8**

Index **9**

Chapter 1

Preface

The *class number* $k(G)$ of a group G is the number of conjugacy classes of G. In 1903, Landau proved in [Lan03] that for every $n \in \mathbb{N}$, there are only finitely many finite groups with exactly n conjugacy classes. The SMALLCLASSNR package provides access to the finite groups with class number at most 14.

These groups were classified in the following papers:

- $k(G) \leq 5$, by Miller in [Mil11] and independently by Burnside in [Bur11]
- $k(G) = 6, 7$, by Poland in [Pol68]
- $k(G) = 8$, by Kosvintsev in [Kos74]
- $k(G) = 9$, by Odincov and Starostin in [OS76]
- $k(G) = 10, 11$, by Vera López and Vera López in [VLVL85] (1)
- $k(G) = 12$, by Vera López and Vera López in [VLVL86] (2)
- $k(G) = 13, 14$, by Vera López and Sangroniz in [VLS07]

(1) In [VLVL85], three distinct groups of the form $(C_5 \times C_5) \rtimes C_4$ order 100 with class number 10 are given. However, only two such groups exist, being the ones with `IdClassNr` equal to `[10,25]` and `[10,26]`.

(2) In [VLVL86], only 48 groups with class number 12 are listed. The three missing groups are provided in the appendix of [VLS07]. These are the groups with `IdClassNr` equal to `[12,13]`, `[12,16]` and `[12,39]`.

Chapter 2

The Small Class Number Library

2.1 Functions

2.1.1 SmallClassNrGroup

▷ `SmallClassNrGroup(id)` (function)

Returns the `i`-th finite group of class number `k` in the library. Alternatively, the pair `[ k, i]` can be given as a single argument `id`. If the group is solvable, it is given as a PcGroup whose Pcgs is a SpecialPcgs. If the group is not solvable, it will be given as a permutation group of minimal permutation degree and with a minimal generating set.

Example
```
gap> G := SmallClassNrGroup( 6, 4 );
<pc group of size 18 with 3 generators>
gap> NrConjugacyClasses( G );
6
gap> IsDihedralGroup( G );
true
```

2.1.2 SmallClassNrGroupsAvailable

▷ `SmallClassNrGroupsAvailable(k)` (function)

Returns `true` if the finite groups of class number `k` are available in the library, and `false` otherwise.

Example
```
gap> SmallClassNrGroupsAvailable( 14 );
true
gap> SmallClassNrGroupsAvailable( 15 );
false
```

2.1.3 AllSmallClassNrGroups

▷ `AllSmallClassNrGroups(arg)` (function)

Returns all finite groups with certain properties as specified by *arg*. The arguments must come in pairs consisting of a function and a value (or list of possible values). At least one of the functions must be `NrConjugacyClasses`. Missing functions will be interpreted as `NrConjugacyClasses`, missing values as `true`.

Example
```
gap> L1 := AllSmallClassNrGroups( [3..5], IsNilpotent );
[ <pc group of size 3 with 1 generator>,
  <pc group of size 4 with 2 generators>,
  <pc group of size 4 with 2 generators>,
  <pc group of size 5 with 1 generator>,
  <pc group of size 8 with 3 generators>,
  <pc group of size 8 with 3 generators> ]
gap> List( L1, NrConjugacyClasses );
[ 3, 4, 4, 5, 5, 5 ]
gap> L2 := AllSmallClassNrGroups( IsSolvable, true, NrConjugacyClasses, 6 );
[ <pc group of size 6 with 2 generators>,
  <pc group of size 12 with 3 generators>,
  <pc group of size 12 with 3 generators>,
  <pc group of size 18 with 3 generators>,
  <pc group of size 18 with 3 generators>,
  <pc group of size 36 with 4 generators>,
  <pc group of size 72 with 5 generators> ]
gap> ForAll( L2, G -> IsSolvable( G ) and NrConjugacyClasses( G ) = 6 );
true
```

2.1.4 OneSmallClassNrGroup

▷ `OneSmallClassNrGroup(`*arg*`)` (function)

Returns one finite group with certain properties as specified by *arg*. The arguments must come in pairs consisting of a function and a value (or list of possible values). At least one of the functions must be `NrConjugacyClasses`. Missing functions will be interpreted as `NrConjugacyClasses`, missing values as `true`.

Example
```
gap> H := OneSmallClassNrGroup( 6, IsAbelian );
<pc group of size 6 with 2 generators>
gap> IsCyclic( H );
true
gap> K := OneSmallClassNrGroup( 10, IsSolvable, true, IsNilpotent, false );
<pc group of size 28 with 3 generators>
gap> NrConjugacyClasses( K ) = 10 and IsSolvable( K ) and not IsNilpotent( K );
true
```

2.1.5 IteratorSmallClassNrGroups

▷ `IteratorSmallClassNrGroups(`*arg*`)` (function)

Returns an iterator that iterates over the finite groups with properties as specified by *arg*. The arguments must come in pairs consisting of a function and a value (or list of possible values). At least one of the functions must be `NrConjugacyClasses`. Missing functions will be interpreted as `NrConjugacyClasses`, missing values as `true`.

Example
```
gap> iter := IteratorSmallClassNrGroups( IsSolvable, false, 11 );
<iterator>
gap> for G in iter do Print( Size( G ), "\n" ); od;
336
720
720
1344
1344
1512
2448
29120
```

2.1.6 NrSmallClassNrGroups

▷ `NrSmallClassNrGroups(k)` (function)

Returns the number of finite groups with class number *k*.

Example
```
gap> NrSmallClassNrGroups( 14 );
92
```

2.1.7 IdClassNr (for IsGroup)

▷ `IdClassNr(k)` (attribute)

Returns the SMALLCLASSNR ID of *G*, i.e. a pair `[k, i]` such that *G* is isomorphic to `SmallClassNrGroup( k, i )`.

Example
```
gap> IdClassNr( AlternatingGroup( 5 ) );
[ 5, 8 ]
gap> A := SmallClassNrGroup( 5, 8 );
Group([ (1,2,3), (1,4,5) ])
gap> IsAlternatingGroup( A );
true
```

References

[Bur11] William Burnside. *Theory of groups of finite order*. The University Press, second edition, 1911. 4

[Kos74] L. F. Kosvintsev. Over the theory of groups with properties given over the centralizers of involutions. *Sverdlovsk (Ural.), Summary thesis Doct*, 1974. 4

[Lan03] Edmund Landau. Über die Klassenzahl der binären quadratischen Formen von negativer Discriminante. *Math Ann*, 56(4):671–676, 1903. 4

[Mil11] George Abram Miller. Groups involving only a small number of sets of conjugate operators. *Arch. der Math. u. Phys.*, 17:199–204, 1911. 4

[OS76] V. A. Odincov and A. I. Starostin. Finite groups with 9 classes of conjugate elements. *Ural. Gos. Univ. Mat. Zap*, 10:114–134, 1976. 4

[Pol68] John Poland. Finite Groups with a given Number of Conjugate Classes. *Canadian J Math*, 20:456–464, 1968. 4

[VLS07] Antonio Vera López and Josu Sangroniz. The finite groups with thirteen and fourteen conjugacy classes. *Math Nachr*, 280(5-6):676–694, 2007. 4

[VLVL85] Antonio Vera López and Juan Vera López. Classification of finite groups according to the number of conjugacy classes. *Isr J Math*, 51(4):305–338, 1985. 4

[VLVL86] Antonio Vera López and Juan Vera López. Classification of finite groups according to the number of conjugacy classes II. *Isr J Math*, 56(2):188–221, 1986. 4

Index

AllSmallClassNrGroups, 5

IdClassNr
  for IsGroup, 7
IteratorSmallClassNrGroups, 6

NrSmallClassNrGroups, 7

OneSmallClassNrGroup, 6

SmallClassNrGroup, 5
SmallClassNrGroupsAvailable, 5